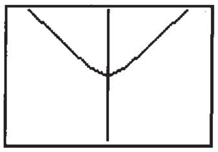
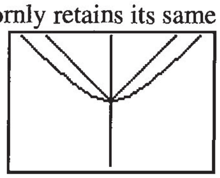
2. Zooming in on the graph of g at the point (0, 1) begins to reveal a smooth turning point. This graph shows the result of three zooms, each by a factor of 4 horizontally and vertically, starting with the window. [–4, 4] by [–1.624, 3.624].



[–0.0625, 0.0625] by [0.959, 1.041]

3. On our grapher, the graph became horizontal after 8 zooms. Results can vary on different machines.

4. As we zoom in on the graphs of f and g together, the differentiable function gradually straightens out to resemble its tangent line, while the nondifferentiable function stubbornly retains its same shape.



[–0.03125, 0.03125] by [0.9795, 1.0205]

Exploration 2 Looking at the Symmetric Difference Quotient Analytically

1. $\dfrac{f(10+h)-f(10)}{h}=\dfrac{(10.01)^2-10^2}{0.01}=20.01$

$f'(10)=2\cdot 10=20$

The difference quotient is 0.01 away from $f'(10)$.

2. $\dfrac{f(10+h)-f(10-h)}{2h}=\dfrac{(10.01)^2-(9.99)^2}{0.02}$

$=20$

The symmetric difference quotient exactly equals $f'(10)$.

3. $\dfrac{f(10+h)-f(10)}{h}=\dfrac{(10.01)^3-10^3}{0.01}$

$=300.3001$

$f'(10)=3\cdot 10^2=300$

The difference quotient is 0.3001 away from $f'(10)$.

$\dfrac{f(10+h)-f(10-h)}{2h}=\dfrac{(10.01)^3-(9.99)^3}{0.02}$

$=300.0001.$

The symmetric difference quotient is 0.0001 away from $f'(10)$.

Quick Review 3.2

1. Yes

2. No (The $f(h)$ term in the numerator is incorrect.)

3. Yes

4. Yes

5. No (The denominator for this expression should be $2h$).

6. All reals

7. $[0, \infty)$

8. $[3, \infty)$

9. The equation is equivalent to $y=3.2x+(3.2\pi+5)$, so the slope is 3.2.

10. $\dfrac{f(3+0.001)-f(3-0.001)}{0.002}$

$=\dfrac{5(3+0.001)-5(3-0.001)}{0.002}$

$=\dfrac{5(0.002)}{0.002}$

$=5$

Section 3.2 Exercises

1. Left-hand derivative:

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{h^2-0}{h}=\lim_{h\to 0^-}h=0$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{h-0}{h}=\lim_{h\to 0^+}1=1$$

Since $0\neq 1$, the function is not differentiable at the point P.

2. Left-hand derivative:

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{2-2}{h}=\lim_{h\to 0^-}0=0$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{2(1+h)-2}{h}$$
$$=\lim_{h\to 0^+}2$$
$$=2$$

Since $0\neq 2$, the function is not differentiable at the point P.

3. Left-hand derivative:

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0^-}\frac{\sqrt{1+h}-1}{h}$$
$$=\lim_{h\to 0^-}\frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)}$$
$$=\lim_{h\to 0^-}\frac{(1+h)-1}{h(\sqrt{1+h}+1)}$$
$$=\lim_{h\to 0^-}\frac{1}{\sqrt{1+h}+1}=\frac{1}{2}$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{[2(1+h)-1]-1}{h}$$
$$=\lim_{h\to 0^+}\frac{2h}{h}$$
$$=\lim_{h\to 0^+}2=2$$

Since $\frac{1}{2}\neq 2$, the function is not differentiable at the point P.

4. Left-hand derivative:

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{(1+h)-1}{h}$$
$$=\lim_{h\to 0^-}1$$
$$=1$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{\frac{1}{1+h}-1}{h}$$
$$=\lim_{h\to 0^+}\frac{1-(1+h)}{h(1+h)}$$
$$=\lim_{h\to 0^+}\frac{-h}{h(1+h)}$$
$$=\lim_{h\to 0^+}-\frac{1}{1+h}$$
$$=-1$$

Since $1\neq -1$, the function is not differentiable at the point P.

5. **(a)** All points in $[-3, 2]$

(b) None

(c) None

6. **(a)** All points in $[-2, 3]$

(b) None

(c) None

7. **(a)** All points in $[-3, 3]$ except $x=0$

(b) None

(c) $x=0$

8. **(a)** All points in $[-2, 3]$ except $x=-1, 0, 2$

(b) $x=-1$

(c) $x=0, x=2$

9. **(a)** All points in $[-1, 2]$ except $x=0$

(b) $x=0$

(c) None

10. **(a)** All points in $[-3, 3]$ except $x=-2, 2$

(b) $x=-2, x=2$

(c) None

11. Since $\lim_{x\to 0}\tan^{-1}x=\tan^{-1}0=0\neq y(0)$, the problem is a discontinuity.

12.

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{h^{4/5}}{h}$$
$$=\lim_{h\to 0^-}\frac{1}{h^{1/5}}$$
$$=-\infty$$

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{h^{4/5}}{h}$$
$$=\lim_{h\to 0^+}\frac{1}{h^{1/5}}$$
$$=\infty$$

The problem is a cusp.

13. Note that

$$y=x+\sqrt{x^2}+2$$
$$=x+|x|+2$$
$$=\begin{cases}2, & x\le 0\\ 2x+2, & x>0.\end{cases}$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{2-2}{h}$$
$$=\lim_{h\to 0^-}0$$
$$=0$$

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{(2h+2)-2}{h}$$
$$=\lim_{h\to 0^+}2$$
$$=2$$

The problem is a corner.

14. $$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{\left(3-\sqrt[3]{h}\right)-3}{h}$$
$$=\lim_{h\to 0}-\frac{\sqrt[3]{h}}{h}$$
$$=\lim_{h\to 0}\left(-\frac{1}{h^{2/3}}\right)$$
$$=-\infty$$

The problem is a vertical tangent.

15. Note that $y=3x-2|x|-1=\begin{cases}5x-1, & x\le 0\\ x-1, & x>0\end{cases}$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{(5h-1)-(-1)}{h}$$
$$=\lim_{h\to 0^-}5$$
$$=5$$

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{(h-1)-(-1)}{h}$$
$$=\lim_{h\to 0^+}1$$
$$=1$$

The problem is a corner.

16. $$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{\sqrt[3]{|h|}-0}{h}$$
$$=\lim_{h\to 0^-}\frac{-\sqrt[3]{h}}{h}$$
$$=\lim_{h\to 0^-}\left(-\frac{1}{h^{2/3}}\right)$$
$$=-\infty$$

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{\sqrt[3]{|h|}-0}{h}$$
$$=\lim_{h\to 0^+}\frac{\sqrt[3]{h}}{h}$$
$$=\lim_{h\to 0^+}\frac{1}{h^{2/3}}$$
$$=\infty$$

The problem is a cusp.

17. $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = \frac{4(0.001)-(0.001)^2-(4(-0.001)-(-0.001)^2)}{0.002}$
$= 4,$ yes it is differentiable.

18. $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = \frac{4(3.001)-(3.001)^2-(4(2.999)-(2.999)^2}{0.002}$
$= -2,$ yes it is differentiable.

19. $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = \frac{4(1.001)+(1.001)^2-4(0.999)-(0.99)^2}{0.002}$
$= 2,$ yes it is differentiable.

20. $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = \frac{(0.001)^3-4(0.001)-((-0.001)^3-4(-0.001))}{0.002}$
$= -3.999999,$ yes it is differentiable.

21. $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = \frac{(-1.999)^3-4(-1.999)-((-2.001)^3-4(-2.001))}{0.002}$
$= 8.000001,$ yes it is differentiable.

22. $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = \frac{(2.001)^3-4(2.001)-((1.999)^3-4(1.999))}{0.002}$
$= -8.000001,$ yes it is differentiable.

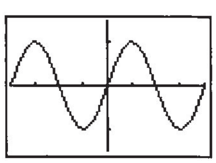
23. $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = \frac{(0.001)^{2/3}-(-0.001)^{2/3}}{0.002}$
$= 0,$ no it is not differentiable. (CUSP)

24. $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = \frac{|3.001-3|-|2.999-3|}{0.002}$
$= 0,$ no it is not differentiable. (CORNER)

25. $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = \frac{(0.001)^{2/5}-(-0.001)^{2/5}}{0.002}$
$= 0,$ no it is not differentiable. (CUSP)

26. $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = \frac{(0.001)^{4/5}-(-0.001)^{4/5}}{0.002}$
$= 0,$ no it is not differentiable. (CUSP)

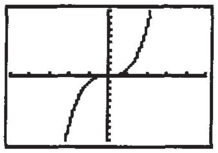
27.



$[-2\pi, 2\pi]$ by $[-1.5, 1.5]$

$\frac{dy}{dx} = \sin x$

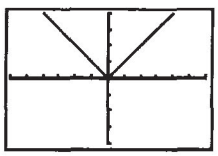
28.



[−5, 5] by [−10, 10]

$\frac{dy}{dx} = x^3$

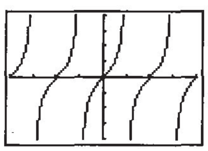
29.



[−6, 6] by [−4, 4]

$\frac{dy}{dx} = \text{abs } (x) \text{or } |x|$

30.



[−2π, 2π] by [−4, 4]

$\frac{dy}{dx} = \tan x$

Note: Due to the way NDER is defined, the graph of $y = \text{NDER } (x)$ actually has two asymptotes for each asymptote of $y = \tan x$. The asymptotes of $y = \text{NDER } (x)$ occur at

$x = \frac{\pi}{2} + k\pi \pm 0.001,$ where k is an integer. A good window for viewing this behavior is [1.566, 1.576] by [−1000, 1000].

31. Find the zeros of the denominator.

$x^2 - 4x - 5 = 0$

$(x+1)(x-5) = 0$

$x = -1$ or $x = 5$

The function is a rational function, so it is differentiable for all x in its domain: all reals except $x = -1, 5$.

32. The function is differentiable except possibly where $3x - 6 = 0$, that is, at $x = 2$. We check for differentiability at $x = 2$, using k instead of the usual h, in order to avoid confusion with the function $h(x)$.

$$\lim_{k \to 0} \frac{h(2+k) - h(2)}{k}$$
$$= \lim_{k \to 0} \frac{[\sqrt[3]{3(2+k) - 6} + 5] - 5}{k}$$
$$= \lim_{k \to 0} \frac{\sqrt[3]{3k}}{k}$$
$$= \sqrt[3]{3} \lim_{k \to 0} \frac{1}{k^{2/3}}$$
$$= \infty$$

The function has a vertical tangent at $x = 2$. It is differentiable for all reals except $x = 2$.

33. Note that the sine function is odd, so

$$P(x) = \sin(|x|) - 1 = \begin{cases} -\sin x - 1, & x < 0 \\ \sin x - 1, & x \ge 0. \end{cases}$$

The graph of $P(x)$ has a corner at $x = 0$. The function is differentiable for all reals except $x = 0$.

34. Since the cosine function is even, $Q(x) = 3\cos(|x|) = 3\cos x$. The function is differentiable for all reals.

35. The function is piecewise-defined in terms of polynomials, so it is differentiable everywhere except possibly at $x = 0$ and at $x = 3$. Check $x = 0$:

$$\lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^-} \frac{(h+1)^2 - 1}{h}$$
$$= \lim_{h \to 0^-} \frac{h^2 + 2h}{h}$$
$$= \lim_{h \to 0^-} (h+2)$$
$$= 2$$

$$\lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^+} \frac{(2h+1) - 1}{h}$$
$$= \lim_{h \to 0^+} 2$$
$$= 2$$

The function is differentiable at $x = 0$.

Check $x = 3$:

Since $g(3) = (4 - 3)^2 = 1$ and $\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-} (2x+1) = 2(3) + 1 = 7$, the function is not continuous (and hence not differentiable) at $x = 3$. The function is differentiable for all reals except $x = 3$.

36. Note that $C(x) = x|x| = \begin{cases} -x^2, & x<0 \\ x^2, & x \geq 0 \end{cases}$, so it is

Check $x = 0$:

$$\lim_{h\to 0} \frac{C(0+h)-C(0)}{h} = \lim_{h\to 0} \frac{h|h|-0}{h} = \lim_{h\to 0} |h| = 0$$

The function is differentiable for all reals.

37. The function $f(x)$ does not have the intermediate value property. Choose some a in $(-1, 0)$ and b in $(0, 1)$. Then $f(a) = 0$ and $f(b) = 1$, but f does not take on any value between 0 and 1. Therefore, by the Intermediate Value Theorem for Derivatives, f cannot be the derivative of any function on $[-1, 1]$.

38. **(a)** $x = 0$ is not in their domains, or, they are both discontinuous at $x = 0$.

(b) For $\frac{1}{x}$: $\text{NDER}\left(\frac{1}{x}, 0\right) = 1{,}000{,}000$

For $\frac{1}{x^2}$: $\text{NDER}\left(\frac{1}{x^2}, 0\right) = 0$

(c) It returns an incorrect response because even though these functions are not defined at $x = 0$, they are defined at $x = \pm 0.001$. The responses differ from each other because $\frac{1}{x^2}$ is even (which automatically makes $\text{NDER}\left(\frac{1}{x^2}, 0\right) = 0$) and $\frac{1}{x}$ is odd.

39. **(a)** $\lim_{x\to 1^-} f(x) = f(1)$

$\lim_{x\to 1^-} (3-x) = a(1)^2 + b(1)$

$2 = a + b$

The relationship is $a + b = 2$.

(b) Since the function needs to be continuous, we may assume that $a + b = 2$ and $f(1) = 2$.

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{(3-(1+h))-2}{h} = \lim_{h\to 0^-} (-1) = -1$$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{a(1+h)^2+b(1+h)-2}{h} = \lim_{h\to 0^+} \frac{a+2ah+ah^2+b+bh-2}{h} = \lim_{h\to 0^+} \frac{2ah+ah^2+bh+(a+b-2)}{h} = \lim_{h\to 0^+} (2a+ah+b) = 2a+b$$

Therefore, $2a + b = -1$. Substituting $2 - a$ for b gives $2a + (2 - a) = -1$, so $a = -3$. Then $b = 2 - a = 2 - (-3) = 5$. The values are $a = -3$ and $b = 5$.

40. True. See Theorem 1.

41. False. The function $f(x) = |x|$ is continuous at $x = 0$ but is not differentiable at $x = 0$.

42. B

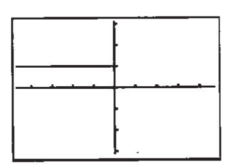
43. A; $\text{NDER}(f, x, a) = \frac{f(a+h)-f(a-h)}{2h} = \frac{\sqrt[3]{1.001-1} - \sqrt[3]{0.999-1}}{0.002} = 100$

The symmetric difference quotient gets larger as h gets smaller, so $f'(1)$ is undefined.

44. B; $\lim_{h\to 0^-} \frac{2(0+h)+1-(2(0)+1)}{h} = \lim_{h\to 0^-} \frac{2h}{h} = 2$

45. C; $\lim_{h\to 0^+} \frac{(0+h)^2+1-(0^2+1)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = 0$

46. **(a)**



[−4.7, 4.7] by [−3.1, 3.1]