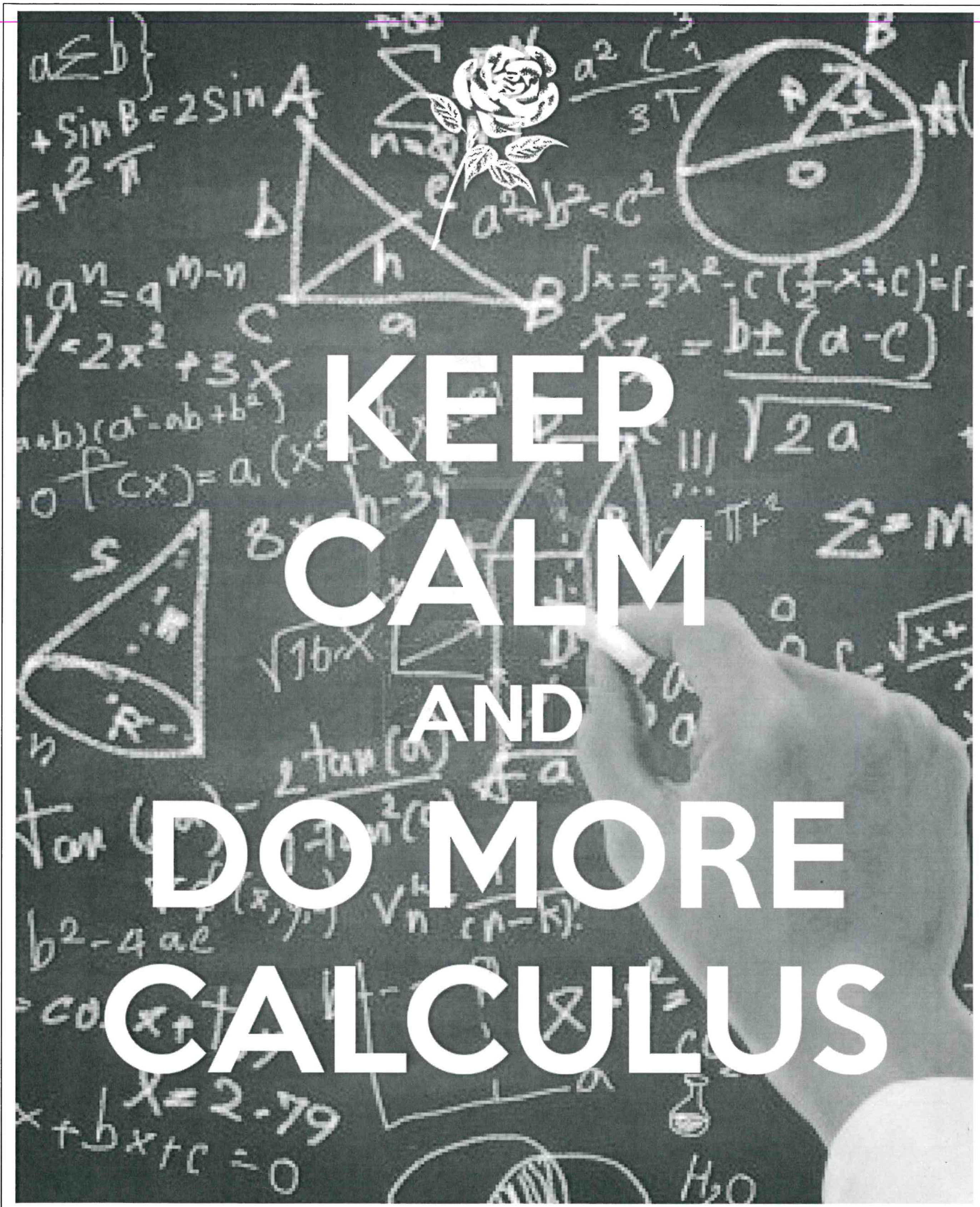
A Review of Introduction to Calculus

Chapters 2 and 3

(plus a couple of new ideas ☺)

Kayed



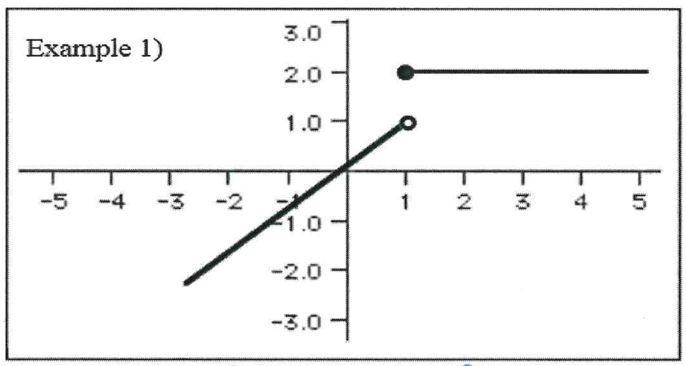
2.1: Evaluating Limits Graphically

The informal definition of a limit is "what is happening to *y* as *x* gets close to a certain number."
In order for a limit to exist, we must be approaching the same *y*-value as we approach some value *c* from either the left or the right side. If this does not happen, we say that the limit does not exist (DNE) as we approach *c*.

If we want the limit of *f* (x) as we approach some value of c from the left hand side, we will write $\lim_{x \to c^-} f(x)$.

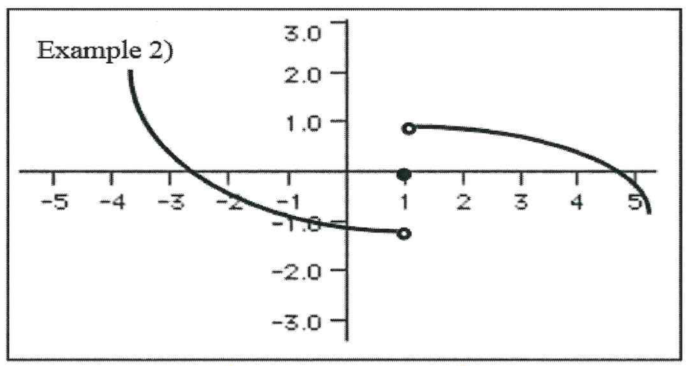
If we want the limit of *f* (x) as we approach some value of c from the right hand side, we will write $\lim_{x \to c^+} f(x)$.

In order for a limit to exist at c, $\lim_{x \to c^-} f(x)$ must equal $\lim_{x \to c^+} f(x)$ and we say $\lim_{x \to c} f(x) = L$.



$\lim_{x \to 1^-} f(x) =$ 1 $\lim_{x \to 1^+} f(x) =$ 2

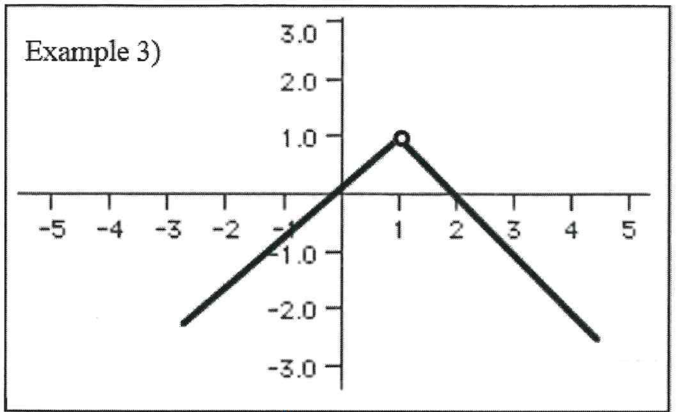
$\lim_{x \to 1} f(x) =$ DNE $f(1) =$ 2



$\lim_{x \to 1^-} f(x) =$ -1 $\lim_{x \to 1^+} f(x) =$ 1

$\lim_{x \to 1} f(x) =$ DNE $f(1) =$ 0

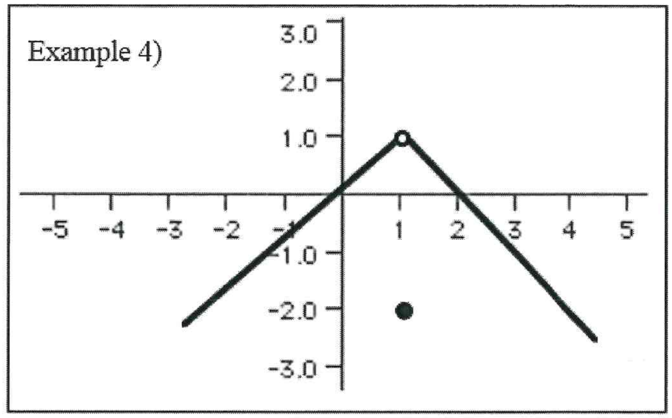
Example 3)



$\lim_{x\to 1^-} f(x) =$ 1 $\quad \lim_{x\to 1^+} f(x) =$ 1

$\lim_{x\to 1} f(x) =$ 1 $\quad f(1) =$ undefined

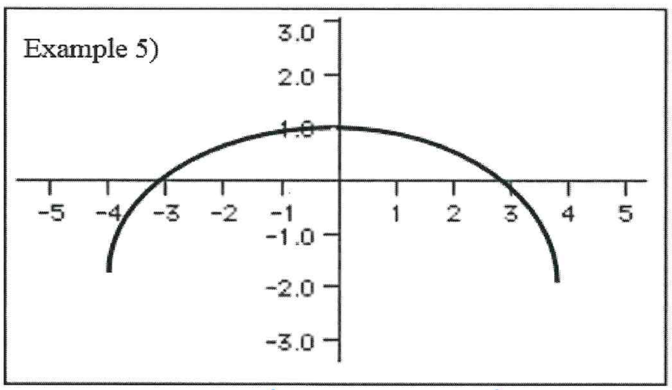
Example 4)



$\lim_{x\to 1^-} f(x) =$ 1 $\quad \lim_{x\to 1^+} f(x) =$ 1

$\lim_{x\to 1} f(x) =$ 1 $\quad f(1) =$ -2

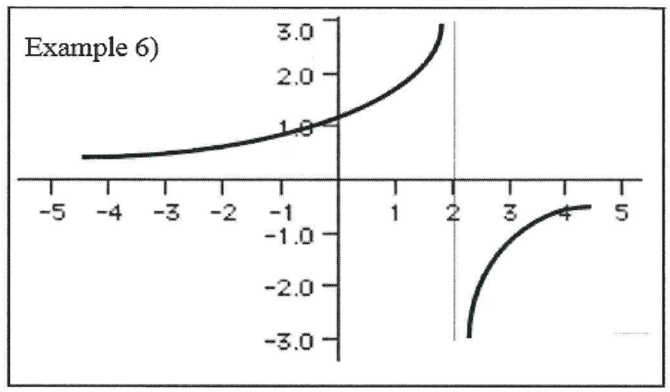
Example 5)



$\lim_{x\to 0^-} f(x) =$ 1 $\quad \lim_{x\to 0^+} f(x) =$ 1

$\lim_{x\to 0} f(x) =$ 1 $\quad f(0) =$ 1

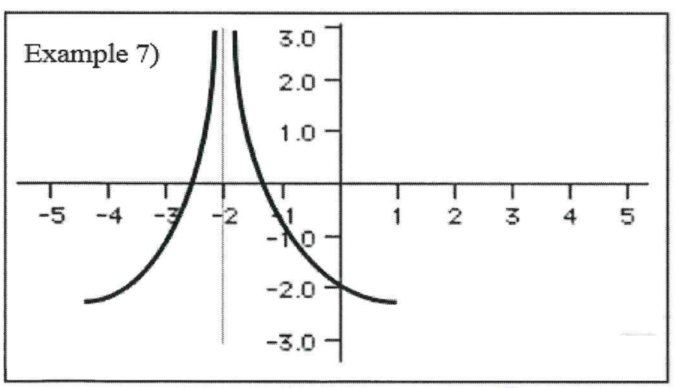
Example 6)



$\lim_{x\to 2^-} f(x) =$ ∞ $\quad \lim_{x\to 2^+} f(x) =$ $-\infty$

$\lim_{x\to 2} f(x) =$ DNE $\quad f(2) =$ undefined

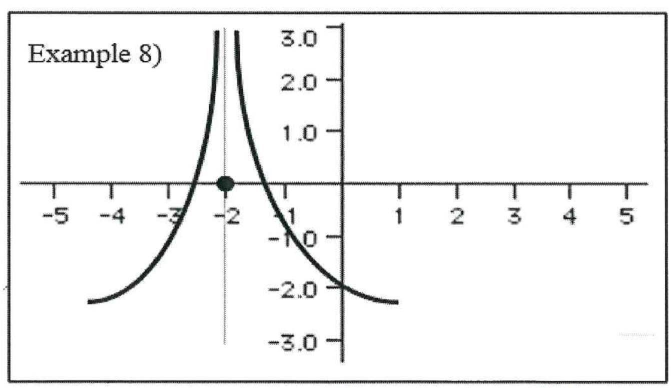
Example 7)



$\lim_{x\to -2^-} f(x) =$ ∞ $\quad \lim_{x\to -2^+} f(x) =$ ∞

$\lim_{x\to -2} f(x) =$ ∞ $\quad f(-2) =$ undefined

Example 8)



$\lim_{x\to -2^-} f(x) =$ ∞ $\quad \lim_{x\to -2^+} f(x) =$ ∞

$\lim_{x\to -2} f(x) =$ ∞ $\quad f(-2) =$ 0

TRY: page 5 in Workbook

2.1: Evaluating Limits Analytically

Properties of Limits

Let *b* and *c* be real numbers and *n* be a positive integer.

Also let *f* and *g* be functions such that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$.

1. **Scalar Multiple**	$\lim_{x \to c} [bf(x)] = bL$
2. **Sum/Difference**	$\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. **Product**	$\lim_{x \to c} [f(x) \cdot g(x)] = L \cdot K$
4. **Quotient**	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$
5. **Power**	$\lim_{x \to c} [f(x)]^n = L^n$

Basic Limits

Let *b* and *c* be real numbers and *n* be a positive integer.

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$

Example: Evaluate each of the following limits

1) $\lim_{x \to 3} 5 = 5$

2) $\lim_{x \to 2} (4x^2 + 3) = 4(2)^2 + 3$

$= 19$

3) $\lim_{x\to -3} \frac{x^2+x-6}{x+3}$

$\lim_{x\to -3} \frac{(x+3)(x-2)}{(x+3)}$

$\lim_{x\to -3} x-2$

$= -3-5$

$= -5$

4) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$

$\lim_{x\to 0} \frac{x+1-1}{x(\sqrt{x+1}+1)}$

$\lim_{x\to 0} \frac{1}{\sqrt{x+1}+1}$

$\frac{1}{2}$

5) $\lim_{x\to 0} \frac{\frac{1}{x+3}-\frac{1}{3}}{x} \cdot \frac{3(x+3)}{3(x+3)}$

$\lim_{x\to 0} \frac{3-(x+3)}{x(3)(x+3)}$

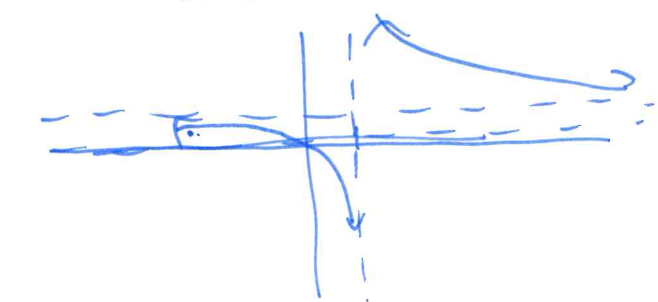
$\lim_{x\to 0} \frac{-\cancel{x}}{\cancel{x}(3)(x+3)}$

$= -\frac{1}{9}$

6) $\lim_{x\to 1} \frac{x}{x-1}$

DNE

Think graphically



$\lim_{x\to 1^+} \frac{x}{x-1} = \infty$

$\lim_{x\to 1^-} \frac{x}{x-1} = -\infty$

$\therefore \lim_{x\to 1} \frac{x}{x-1}$ DNE

OR numerically.

Substitute x values from the left and right of 1.

TRY: pages 6 and 7 in Workbook

2.2: Limits as $x \to \pm\infty$

We start this topic with a very important and rather straight-forward theorem.

Limits at Infinity Theorem

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0 \quad \text{and when possible,} \quad \lim_{x\to-\infty} \frac{c}{x^r} = 0$$

Example 1

Find the limit $\lim_{x\to\infty}\left(5 - \frac{2}{x^2}\right)$

$$\lim_{x\to\infty} 5 - \lim_{x\to\infty}\left(\frac{-2}{x^2}\right)$$

$$5 - 0$$

$$5$$

Example 2

Find the limit $\lim_{x\to\infty}\frac{2x-1}{x+1}$

Divide by the highest power of x.

$$\lim_{x\to\infty} \frac{\frac{2x}{x} + \frac{1}{x}}{\frac{x}{x} + \frac{1}{x}}$$

$$\lim_{x\to\infty} \frac{2 + \frac{1}{x}}{1 + \frac{1}{x}}$$

$$\frac{2 - 0}{1 + 0}$$

$$2$$

Example 3 A Comparison of Three Rational Functions

a. $\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$

$$\lim_{x\to\infty} \frac{\frac{2x}{x^2}+\frac{5}{x^2}}{\frac{3x^2}{x^2}+\frac{1}{x^2}}$$

$= 0$

b. $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1}$

$$\lim_{x\to\infty} \frac{\frac{2x^2}{x^2}+\frac{5}{x^2}}{\frac{3x^2}{x^2}+\frac{1}{x^2}}$$

$= \frac{2}{3}$

c. $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1}$

$$\lim_{x\to 0} \frac{\frac{2x^3}{x^3}+\frac{5}{x^3}}{\frac{3x^2}{x^3}+\frac{1}{x^3}}$$

$= \frac{2+0}{0}$

DNE

NOTE: Would it have made any difference in either example above if x approached $-\infty$?

No!

Example 4: A Function Where The Results Differ

Find each limit analytically. Then sketch the function on your graphing calculator.

a. $\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

$$\lim_{x\to\infty} \frac{\frac{3x}{x}-\frac{2}{x}}{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}$$

$$\lim_{x\to\infty} \frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}}$$

$$\frac{3-0}{\sqrt{2}}$$

$= \frac{3}{\sqrt{2}}$

b. $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

$$\lim_{x\to-\infty} \frac{\frac{3x}{x}-\frac{2}{x}}{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}$$

$$\lim_{x\to-\infty} \frac{\frac{3x}{x}-\frac{2}{x}}{-\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}$$

$$\lim_{x\to-\infty} \frac{3-\frac{2}{x}}{-\sqrt{2+\frac{1}{x^2}}}$$

$= -\frac{3}{\sqrt{2}}$

Note: $\sqrt{x^2} = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

TRY: page 8 in Workbook

$\sqrt{x^2} = |x|$

Example 3 A Comparison of Three Rational Functions

a. $\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$

$$\lim_{x\to\infty} \frac{\frac{2x}{x^2}+\frac{5}{x^2}}{\frac{3x^2}{x^2}+\frac{1}{x^2}}$$

$= 0$

b. $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1}$

$$\lim_{x\to\infty} \frac{\frac{2\cancel{x^2}}{\cancel{x^2}}+\frac{5}{x^2}}{\frac{3\cancel{x^2}}{\cancel{x^2}}+\frac{1}{x^2}}$$

$= \frac{2}{3}$

c. $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1}$

$$\lim_{x\to\infty} \frac{\frac{2x^3}{x^3}+\frac{5}{x^3}}{\frac{3x^2}{x^3}+\frac{1}{x^3}}$$

DNE

NOTE: Would it have made any difference in either example above if x approached $-\infty$?

Not in the above cases

Example 4: A Function Where The Results Differ

Find each limit analytically. Then sketch the function on your graphing calculator.

a. $\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

Rewrite 1st

factor out x^2

$$\lim_{x\to\infty} \frac{3x-2}{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}$$

$$\lim_{x\to\infty} \frac{3x-2}{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}$$

Recall $\sqrt{x^2} = |x|$

$$\lim_{x\to\infty} \frac{3x-2}{|x|\sqrt{2+\frac{1}{x^2}}}$$

$$\lim_{x\to\infty} \frac{\frac{3\cancel{x}}{\cancel{x}}-\frac{2}{x}}{\frac{+\cancel{(x)}\sqrt{2x+\frac{1}{x^2}}}{\cancel{x}}}$$

$$\frac{3-0}{\sqrt{2-0}} = \frac{3}{\sqrt{2}}$$

b. $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

$$\lim_{x\to-\infty} \frac{3x-2}{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}$$

$$\lim_{x\to-\infty} \frac{3x-2}{|x|\sqrt{2+\frac{1}{x^2}}}$$

$$\lim_{x\to-\infty} \frac{\frac{3\cancel{x}}{\cancel{x}}-\frac{2}{x}}{\frac{-(x)\sqrt{2+\frac{1}{x^2}}}{\cancel{x}}}$$

$$= \frac{3-0}{-\sqrt{2+0}} = -\frac{3}{\sqrt{2}}$$

Recall $|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

TRY: page 8 in Workbook

2.1: The Sandwich Theorem (Something New!)

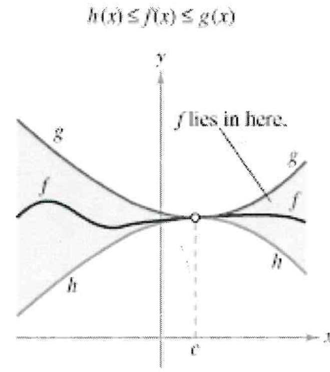
Before we discuss trigonometric limits, we will first familiarize ourselves with "The Sandwich Theorem (Squeeze Theorem)"

The Sandwich Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then $\lim_{x\to c} f(x)$ exists and is equal to L.



Example 1

Show that $\lim_{x\to 0}\left(x^2 \sin\left(\frac{1}{x}\right)\right) = 0$

Since $-1 \le \sin x \le 1$

So we know $x \ne 0$

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$

$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$

Since $\lim_{x\to 0} -x^2 = 0$

$\lim_{x\to 0} x^2 = 0$

$\therefore \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

Example 2

Find $\lim_{x\to\infty} \frac{\sin x}{x}$

Since $-1 \le \sin x \le 1$ $x \ne 0$

for $x>0$ we know $\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

And $\lim_{x\to\infty} \frac{-1}{x} = 0$

$\lim_{x\to\infty} \frac{1}{x} = 0$

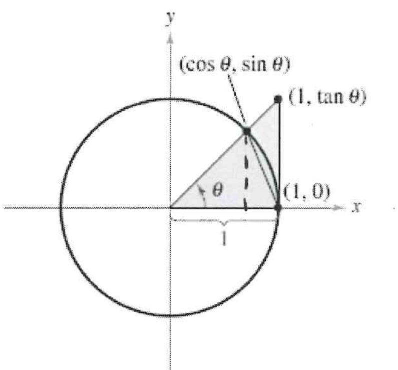
it follows that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

2.1/2.2: Special Trigonometric Limits (Something New!)

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$ or $\lim\limits_{x\to 0}\dfrac{x}{\sin x} = 1$

2. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x} = 0$ $\lim\limits_{x\to 0}\dfrac{\cos x - 1}{x} = 0$

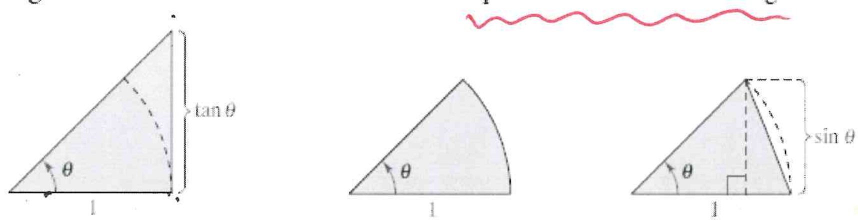
Proof of Special Trig Limit #1 Above



A circular sector is used to prove Theorem 1.9.
Figure 1.22

Note:
Area of sector
$\pi r^2 \left(\dfrac{\theta}{2\pi}\right)$
$= \dfrac{r^2(\theta)}{2}$

PROOF To avoid the confusion of two different uses of x, the proof is presented using the variable θ, where θ is an acute positive angle *measured in radians.* Figure 1.22 shows a circular sector that is squeezed between two triangles.



Area of triangle	$\geq$	Area of sector	$\geq$	Area of triangle
$\dfrac{\tan\theta}{2}$	$\geq$	$\dfrac{\theta}{2}$	$\geq$	$\dfrac{\sin\theta}{2}$

$\dfrac{BH}{2}$

Multiplying each expression by $2/\sin\theta$ produces

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1$$

and taking reciprocals and reversing the inequalities yields

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

$\lim\limits_{x\to 0} \cos\theta = 1$ $\lim\limits_{x\to 0} 1 = 1$ $\therefore \lim\limits_{x\to 0} \dfrac{\sin\theta}{\theta} = 1$

Solving Trig Limits Analytically

The special limits may help us evaluate the following

1) $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$

$\lim\limits_{x\to 0} \dfrac{\sin x}{\cos x}\cdot\dfrac{1}{x}$

$= \lim\limits_{x\to 0} \dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x}$

$= 1 \cdot 1$

$= 1$

2) $\lim\limits_{x\to 0}\dfrac{(\sin 4x)(4)}{x\,(4)}$

$4 \lim\limits_{x\to 0}\left(\dfrac{\sin 4x}{4x}\right)$

$= 4$

3) $\lim_{x \to 0} \frac{1-\cos^2 x}{x}$

$$\lim_{x \to 0} \frac{(1-\cos x)(1+\cos x)}{x}$$

$$\lim_{x \to 0} \frac{1-\cos x}{x} \cdot 1+\cos x$$

$$0(2)$$

$$= 0$$

4) $\lim_{x \to 0} \frac{\sin 7x}{4x}$

$$\frac{1}{4} \lim_{x \to 0} 7 \frac{\sin 7x}{7x}$$

$$\frac{7}{4} \lim_{x \to 0} \frac{\sin 7x}{7x}$$

$$= \frac{7}{4}$$

5) $\lim_{x \to 0} \frac{\sin 3x \cos 2x}{x}$

$$\lim_{x \to 0} 3 \frac{\sin 3x}{3x} \cdot \cos 2x$$

$$3 \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \cos 2x$$

$$3(1)(1)$$

$$= 3$$

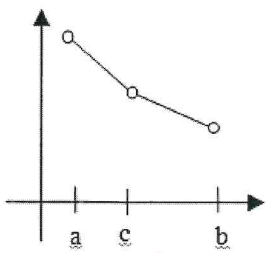
TRY: page 10 and 11 in Workbook

2.3: Continuity

Continuity implies "no interruptions" in the graph.

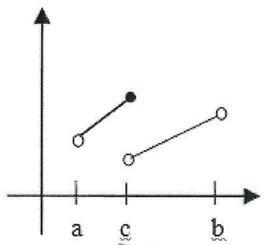
Examples of Discontinuity in a Function over (a, b)

1.



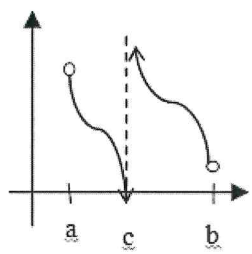
removable

2.



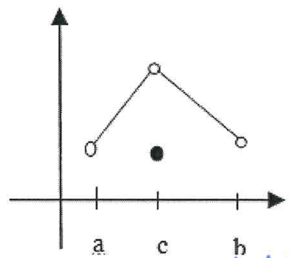
jump.

or **3.**



infinite
or asymptotic.

4.



removable

3 Criteria of Continuity
In order for a function, f(x) to be continuous at ***c***, the following three conditions must be met.
1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

Removable vs Nonremovable Discontinuitites

Removable	Nonremovable
you can fill a hole" Case 1 & 4 above	"you can't fill a hole" Case 2 & 3 above
"a denominator factor that will cancel"	"a denominator factor that will NOT cancel"

Example 1: Discuss the continuity of each.

a. $f(x) = \frac{1}{x}$

$f(0)$ is not defined.

$\therefore$ discontinuous @ $x=0$

Asymptotic Discontinuity

b. $f(x) = \frac{x+4}{x^2 - 2x - 24}$

$f(x) = \frac{\cancel{x+4}}{(x-6)\cancel{(x+4)}}$

Removable discontinuity @ $x = -4$

Asymptotic discontinuity @ $x = 6$

$f(x)$ is undefined @ $x = 6$ and $x = -4$.

c. $h(x) = \begin{cases} x+1 & x \le 0 \\ x^2+1 & x > 0 \end{cases}$

$y = x+1$ and $y = x^2+1$ are continuous functions

check for continuity at $x=0$

$\lim_{x \to 0^+} h(x) = 1$ $\quad h(0) = 1$

$\lim_{x \to 0^-} h(x) = 1$

$\therefore \lim_{x \to 0} h(x) = h(0) = 1$

$h(x)$ is a continuous function.

Example 2: Find the constant, a, such that the function is continuous on the entire real number line. Show a complete analysis of your conclusion.

$$f(x) = \begin{cases} ax+1 & \text{if } x \le 3 \\ ax^2 - 1 & \text{if } x > 3 \end{cases}$$

Since $\lim_{x \to 3} f(x) = f(3)$

$f(3) = 3a+1$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = f(3)$

$3a+1 = 9a-1$

$2 = 6a$

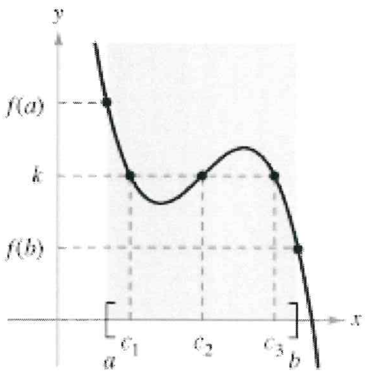
$\frac{1}{3} = a$

TRY: page 12 and 13 in Workbook

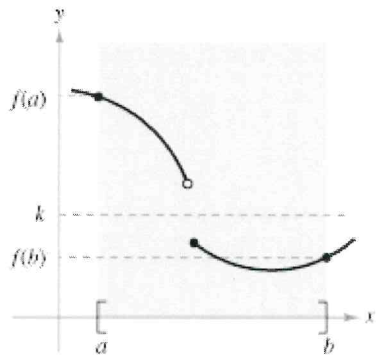
2.3: The Intermediate Value Theorem (Something New!)

The Intermediate Value Theorem

If f is continuous on the closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$, there is at least one number c in $[a,b]$ such that $f(c)=k$.



f is continuous on $[a, b]$.
[There exist three c's such that $f(c) = k$.]



f is not continuous on $[a, b]$.
[There are no c's such that $f(c) = k$.]

Example
Use the Intermediate Value Theorem to show the polynomial function $f(x)=x^3+2x-1$ has a zero on the interval [0,1].

$f(0) = -1$ $\qquad$ $f(1) = 1$

Since $f(x)$ is continous on $[0,1]$ and
$-1 < 0 < 1$ the IVT states
there must exist one c value such that $f(c)=0$
$\therefore$ there is at least one x-intercept.

3.1: The Definition of the Derivative

The derivative of f(x) with respect with respect to x is the function

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Example Use the definition to determine the derivative of $f(x) = 4x^2$

$$f'(x) = \lim_{h\to 0}\frac{4(x+h)^2 - 4x^2}{h}$$

$$= \lim_{h\to 0}\frac{4(x^2+2xh+h^2) - 4x^2}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{4x^2} + 8xh + 4h^2 \cancel{-4x^2}}{h}$$

$$= \lim_{h\to 0} 8x + 4h$$

$$= 8x$$

b) What is the derivative of $f(x) = 4x^2$ when $x = 3$?

$$f'(3) = 8(3)$$
$$= 24$$

c) Determine the equation of the tangent line to $f(x) = 4x^2$ when $x = 3$.

pt $(3, 36)$ $m = 24$

$$y - 36 = 24(x-3)$$

d) Determine the equation of the normal line to $f(x) = 4x^2$ when $x = 3$

$\perp$ to tangent line $\therefore m = \frac{-1}{24}$

$$y - 36 = \frac{-1}{24}(x-3)$$

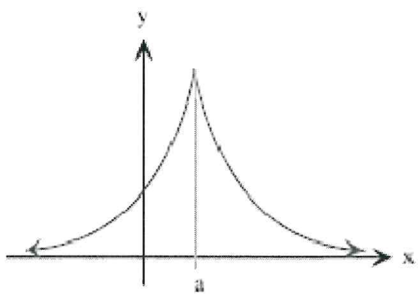
3.2: Differentiability

A function is said to be differentiable at $x = a$ if $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

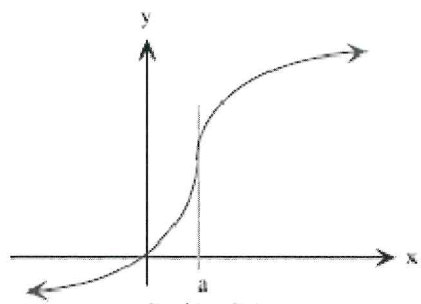
Geometrically:

The derivative will not exist at $x = a$ if the following situations apply

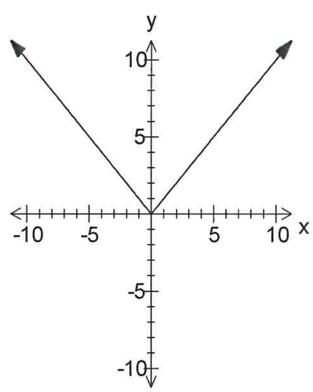
a) the function is discontinuous at *a*
b) the graph has a corner at *a*
c) the graph has a cusp at *a*
d) a graph has a vertical tangent at *a*



$\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ does not exist



$\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ is infinite

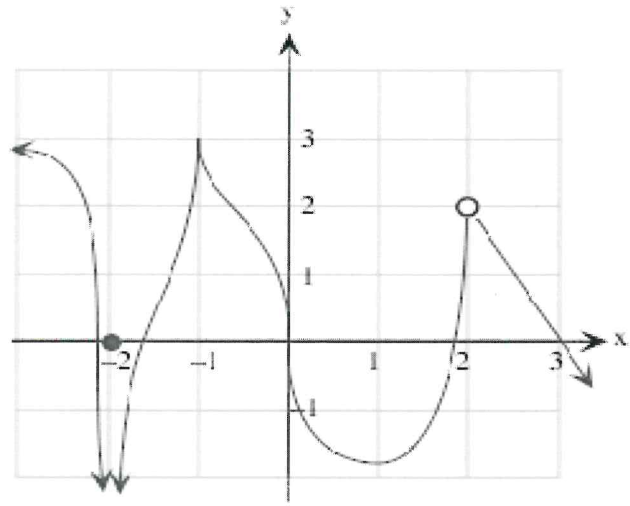


Theorem: Differentiability implies continuity

If a function has a derivative at x = c, it must be continuous at that point.

Example 1

Given the function f



Choose from the following to complete the blanks below

Differentiable, discontinuous, undefined, continuous but not differentiable

a) f is discontinuous at x = -2

b) f is continuous but not dif. at x = -1

c) f is continuous but not dif. at x = 0

d) f is differentiable at x = 1

e) f is undefined at x = 2

f) f is differentiable at x = 3

Example 2

Let $f(x) = \begin{cases} x^2 - 4x + 3, x \le 4 \\ ax + b, x > 4 \end{cases}$

Find the values of a and b such that f(x) is differentiable at x = 4.

f(x) must be continuous @ 4 $\therefore$ $\lim_{x \to 4} f(x)$ must exist

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^-} f(x)$$

$$4a + b = 16 - 16 + 3$$

$$4a + b = 3$$

$$f'(x) = \begin{cases} 2x - 4, & x \le 4 \\ a, & x > 4 \end{cases}$$

$$\lim_{x \to 4^+} f'(x) = \lim_{x \to 4^-} f'(x)$$

$$a = 8 - 4$$

$$a = 4$$

$$\therefore \quad 4a + b = 3$$

$$8 + b = 3$$

$$b = -13.$$

Textbook: p. 114 # 1,3,5,7,9,31,35,38, 40-45