Grade 12
Pre-Calculus Mathematics
Achievement Test

Marking Guide

June 2016

Manitoba

A wheel has a diameter of 20 cm and rotates through a central angle of 252°.
Determine how far the wheel rolled.

Solution

$$\theta = \left(252^\circ\right)\left(\frac{\pi}{180^\circ}\right)$$

$$= \frac{7\pi}{5}$$

1 mark for conversion

$$s = \theta r$$

$$= \left(\frac{7\pi}{5}\right)\left(\frac{20}{2}\right)$$

$$= 14\pi \text{ cm}$$

or

$$= 43.982 \text{ cm}$$

1 mark for substitution

2 marks

Question 2 T5

Solve the following equation over the interval $[0, 2\pi]$:

$$3\sin^2\theta - 10\sin\theta - 8 = 0$$

Solution

$3\sin^2\theta - 10\sin\theta - 8 = 0$

$(3\sin\theta + 2)(\sin\theta - 4) = 0$

$\sin\theta = -\frac{2}{3}$	$\sin\theta = 4$	1 mark for solving for $\sin\theta$ (½ mark for each branch)
$\theta_r = 0.729\ 728$	No solution	2 marks for solving for θ (1 mark for indicating no solution, ½ mark for each value)
$\theta = 3.871$		
$\theta = 5.553$		

3 marks

Determine and simplify the fourth term in the expansion of $\left(2x^4 - 3y\right)^8$.

Solution

$t_4 = {}_8C_3\left(2x^4\right)^5(-3y)^3$ 2 marks (1 mark for ${}_8C_3$, ½ mark for each consistent factor)

$= 56\left(32x^{20}\right)\left(-27y^3\right)$

$= -48\ 384x^{20}y^3$ 1 mark for simplification (½ mark for coefficient, ½ mark for exponents)

3 marks

Question 4 R10

Sheeva's bank is lending her \$50 000 at an annual interest rate of 6%, compounded monthly, to purchase a car.

Given that the last payment will be a partial payment, determine how many full monthly payments of \$800 Sheeva will have to make.

The formula below may be used.

$$PV = \frac{R\left[1-(1+i)^{-n}\right]}{i}$$

where $PV =$ the present value of the amount borrowed

$R =$ the amount of each periodic payment

$i = \dfrac{\text{annual interest rate (as a decimal)}}{\text{the number of compounding periods per year}}$

$n =$ the number of equal periodic payments

Express your answer as a whole number.

Solution

$$50\ 000 = \frac{800\left[1-\left(1+\frac{0.06}{12}\right)^{-n}\right]}{\frac{0.06}{12}}$$ ½ mark for substitution

$$250 = 800\left[1-(1+0.005)^{-n}\right]$$

$$0.3125 = 1-1.005^{-n}$$

$$-0.6875 = -1.005^{-n}$$

$$0.6875 = 1.005^{-n}$$ ½ mark for simplification

$$\log 0.6875 = -n\log 1.005$$ ½ mark for applying logarithms
1 mark for power law

$$\frac{\log 0.6875}{-\log 1.005} = n$$

$$75.125\ 880\ 88 = n$$ ½ mark for solving for n

$\therefore$ 75 full monthly payments are needed

3 marks

Question 5 P1

An employee asked 10 people in an ice cream shop to wait in line.

Determine the number of different arrangements possible if two of the people, Jamie and John, refused to stand next to each other in the line.

Solution

$10! - 9!2!$	½ mark for $10!$ 1 mark for product of $9!2!$ (½ mark for $9!$, ½ mark for $2!$) ½ mark for subtraction
2 903 040	**2 marks**

Question 6 R5

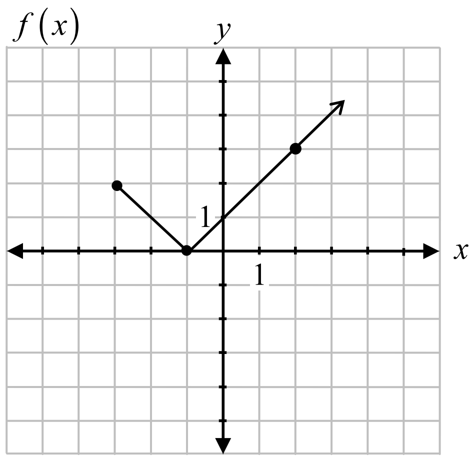
The point $(-2, 4)$ is on the graph of $f(x)$.

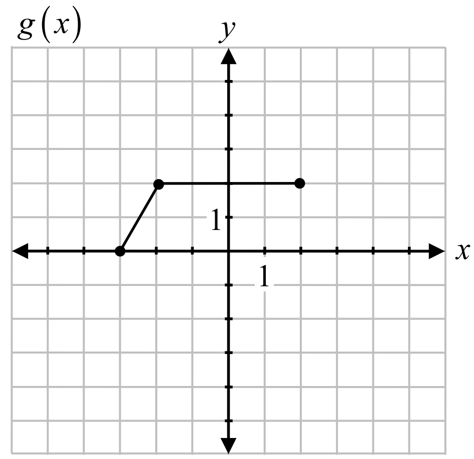
State the coordinates of the corresponding point when $f(x)$ is reflected over the y-axis.

Solution

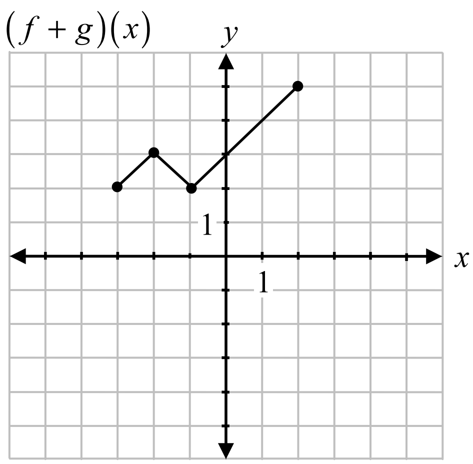
$(2, 4)$ **1 mark**

Given the graphs of $f(x)$ and $g(x)$, sketch the graph of $(f+g)(x)$.





Solution



1 mark for operation of addition
1 mark for restricted domain

2 marks

Question 8 R8

Using the laws of logarithms, fully expand the expression:

$$\log_2\left(\frac{w^3x}{y-1}\right)$$

Solution

$3\log_2 w + \log_2 x - \log_2(y-1)$

1 mark for power law
1 mark for product law
1 mark for quotient law

3 marks

Question 9 T5, T6

Solve the following equation algebraically for θ, where $0 \le \theta \le 2\pi$:

$$2\cos 2\theta = 1$$

Solution

Method 1

$$2\left(2\cos^2\theta - 1\right) = 1$$ 1 mark for substitution of an appropriate identity

$$4\cos^2\theta - 2 = 1$$

$$\cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \pm\frac{\sqrt{3}}{2}$$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\qquad$ $\cos\theta = -\frac{\sqrt{3}}{2}$ $\qquad$ 1 mark for solving for $\cos\theta$ (½ mark for each value)

$\theta = \frac{\pi}{6}, \frac{11\pi}{6}$ $\qquad$ $\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$ $\qquad$ 2 marks (½ mark for each value of θ)

4 marks

Method 2

$$2\left(1 - 2\sin^2\theta\right) = 1$$ 1 mark for substitution of an appropriate identity

$$2 - 4\sin^2\theta = 1$$

$$-4\sin^2\theta = -1$$

$$\sin^2\theta = \frac{1}{4}$$

$$\sin\theta = \pm\frac{1}{2}$$

$\sin\theta = \frac{1}{2}$ $\qquad$ $\sin\theta = -\frac{1}{2}$ $\qquad$ 1 mark for solving for $\sin\theta$ (½ mark for each value)

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ $\qquad$ $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ $\qquad$ 2 marks (½ mark for each value of θ)

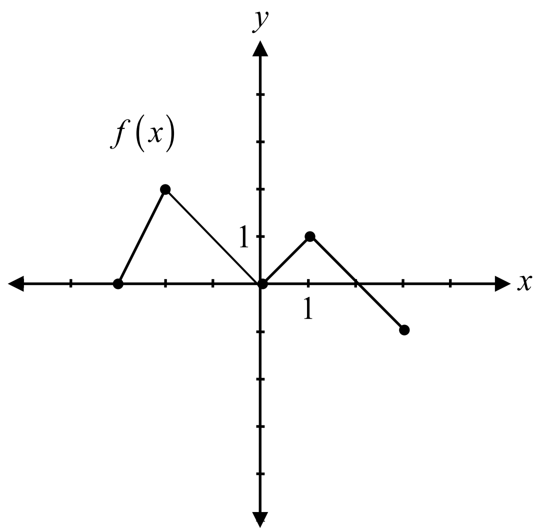
4 marks

Note(s):

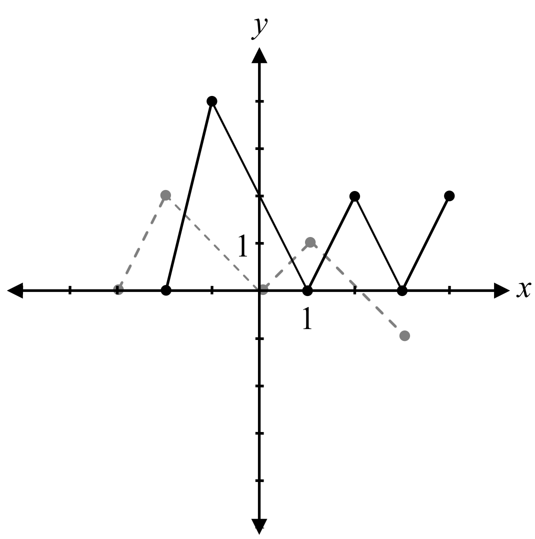
- Deduct a maximum of 1 mark if student omits second branch when taking the square root.

Question 10 R1, R2, R3

Given the graph of $y = f(x)$, sketch the graph of $y = 2\left|f(x-1)\right|$.



Solution



1 mark for vertical stretch
1 mark for horizontal translation
1 mark for absolute value

3 marks

Question 11 — T6

Prove the identity for all permissible values of θ:

$$\cos\theta + \tan\theta\sin\theta = \frac{\tan\theta\sin\theta}{1-\cos^2\theta}$$

Solution

Left-Hand Side	Right-Hand Side	
$\cos\theta + \tan\theta\sin\theta$	$\frac{\tan\theta\sin\theta}{1-\cos^2\theta}$	
$\cos\theta + \frac{\sin\theta}{\cos\theta}\sin\theta$	$\frac{\frac{\sin\theta}{\cos\theta}\bullet\sin\theta}{\sin^2\theta}$	1 mark for correct substitution of identities
$\cos\theta + \frac{\sin^2\theta}{\cos\theta}$	$\frac{\sin^2\theta}{\cos\theta}\bullet\frac{1}{\sin^2\theta}$	1 mark for algebraic strategies
$\frac{\cos^2\theta+\sin^2\theta}{\cos\theta}$	$\frac{1}{\cos\theta}$	1 mark for logical process to prove the identity
$\frac{1}{\cos\theta}$		**3 marks**

Question 12 P1

Raoul has 8 shirts, 5 pairs of pants, and 3 hats. He adds the options together and determines that he has 16 different outfits to wear.

Raoul made an error in calculating the number of different outfits. Describe how to determine the correct number of outfits.

Solution

Raoul should have multiplied the number of clothing items to determine the total number of outfits.

1 mark

Question 13 R1

Given $f(x) = 2x - 1$ and $g(x) = x^2 + 1$:

a) Determine $f(x) \bullet g(x)$.

b) Determine $g(g(x))$.

Solution

a) $f(x) \bullet g(x) = (2x - 1)(x^2 + 1)$ 1 mark for product

$= 2x^3 + 2x - x^2 - 1$ **1 mark**

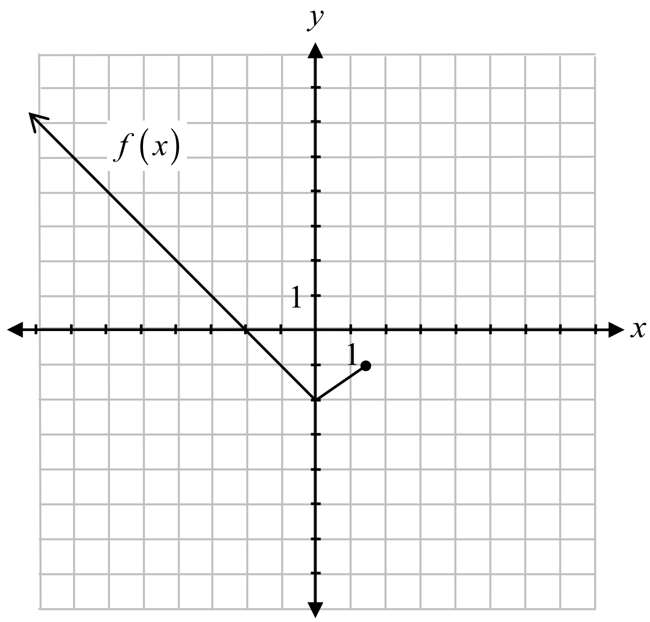
$= 2x^3 - x^2 + 2x - 1$

b) $g(g(x)) = (x^2 + 1)^2 + 1$ 1 mark for composition

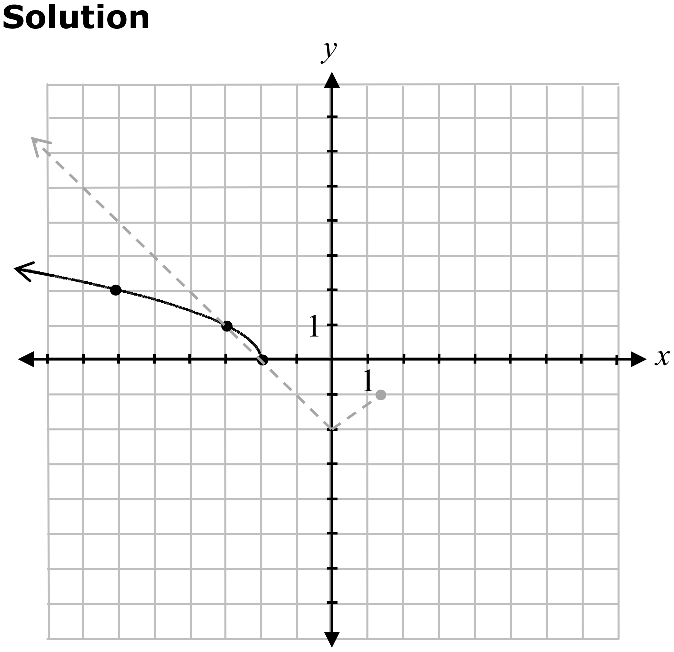
$= x^4 + 2x^2 + 1 + 1$ **1 mark**

$= x^4 + 2x^2 + 2$

Given the graph of $y = f(x)$, sketch the graph of $y = \sqrt{f(x)}$.



Solution



1 mark for restricting the domain
½ mark for shape between invariant points
½ mark for shape to the left of invariant points

2 marks

Answer Key for Selected Response Questions

Question	Answer	Learning Outcome
15	C	R11
16	B	P3
17	D	R13
18	B	R7
19	A	T5
20	B	R13
21	D	R6
22	C	T1

Question 15 R11

Given the polynomial function $P(x) = x^4 - 5x^2 - 2x + 6$, if $P(1) = 0$, identify which statement is true.

a) The y-intercept is 1.

b) $P(x)$ has a factor of $(x+1)$.

c) The graph has a zero at 1

d) The graph has a zero at -1.

Question 16 P3

There are 6 different books that are being distributed evenly amongst three people.
Identify which expression represents the number of possible combinations.

a) ${}_6C_2 \bullet {}_6C_2 \bullet {}_6C_2$

b) ${}_6C_2 \bullet {}_4C_2 \bullet {}_2C_2$

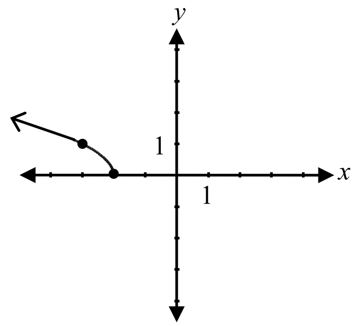
c) ${}_2C_2 \bullet {}_2C_2 \bullet {}_2C_2$

d) $3 \bullet {}_6C_2$

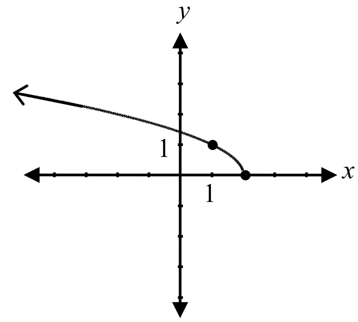
Question 17 R13

Identify the graph that corresponds to the function $f(x) = -\sqrt{(x-2)}$.

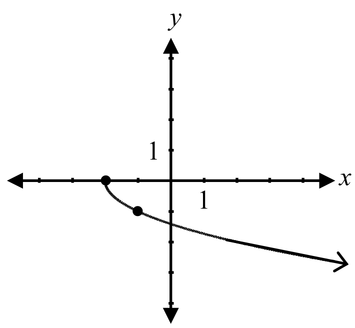
a)



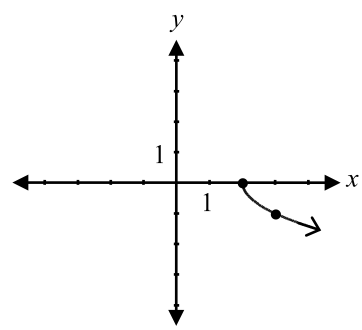
b)



c)



d)



Question 18 R7

Solve:

$$7^{\log_7 2} = x$$

a) $x = 1$

b) $x = 2$

c) $x = 7$

d) $x = 49$

Question 19 T5

Identify the equation that has a general solution of $\left.\begin{array}{l}\theta = \dfrac{\pi}{6} + 2\pi k \\ \theta = \dfrac{5\pi}{6} + 2\pi k\end{array}\right\}$ where $k \in \mathbb{Z}$.

a) $\sin\theta = \dfrac{1}{2}$

b) $\cos\theta = \dfrac{1}{2}$

c) $\sin\theta = \dfrac{\sqrt{3}}{2}$

d) $\cos\theta = \dfrac{\sqrt{3}}{2}$

Question 20 R13

Identify the function that has a domain of $x \leq -2$ and a range of $y \geq 3$.

a) $y = \sqrt{x+2} + 3$

b) $y = \sqrt{-(x+2)} + 3$

c) $y = -\sqrt{x-2} - 3$

d) $y = -\sqrt{-(x-2)} - 3$

Question 21 R6

Given $f(x) = 3x + 2$, identify $f^{-1}(x)$.

a) $f^{-1}(x) = -3x - 2$

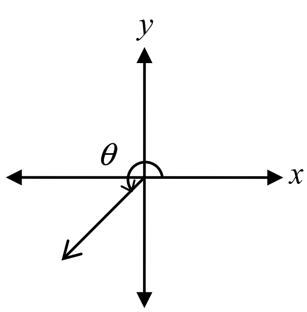
b) $f^{-1}(x) = 2x + 3$

c) $f^{-1}(x) = \frac{x}{3} - 2$

d) $f^{-1}(x) = \frac{x-2}{3}$

Question 22 T1

Identify a possible value for the angle θ sketched in standard position.



a) 2

b) 3

c) 4

d) 5

Question 23 R10

Solve the following equation:

$$\log_3(x+3)+\log_3(x-5)=2$$

Solution

$$\log_3\left[(x+3)(x-5)\right]=2$$ 1 mark for product law

$$(x+3)(x-5)=3^2$$ 1 mark for exponential form

$$x^2-2x-15=9$$

$$x^2-2x-24=0$$

$$(x-6)(x+4)=0$$

$x=6$ ~~$x=-4$~~

½ mark for solving for x
½ mark for rejecting extraneous root

3 marks

Question 24 T1

State a coterminal angle for $\theta = \frac{9\pi}{4}$.

Solution

$$\frac{9\pi}{4} + \frac{8\pi}{4} = \frac{17\pi}{4}$$

1 mark

or

$$405^\circ - 360^\circ = 45^\circ$$

Note(s):

- Other answers are possible.

Sketch the graph of the function $f(x) = \dfrac{2x+2}{x^2-1}$.

Solution

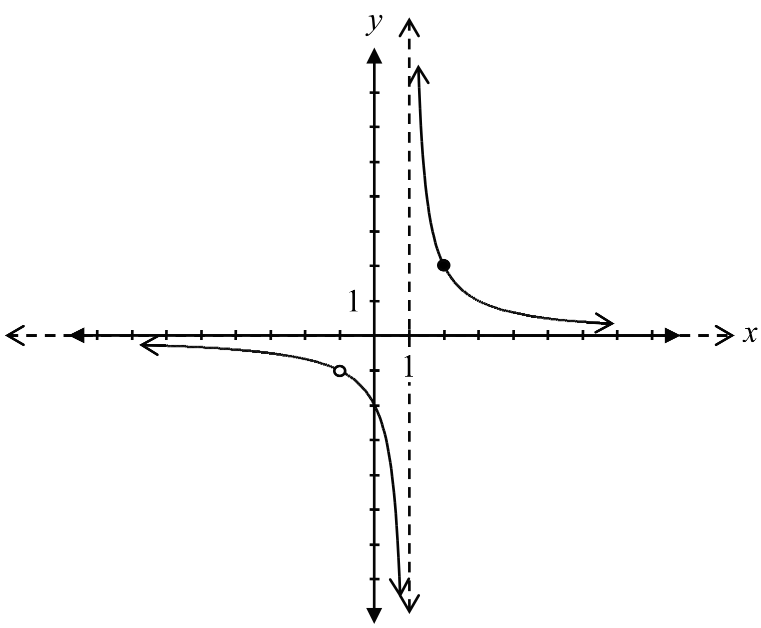
$$f(x) = \frac{2\cancel{(x+1)}}{(x-1)\cancel{(x+1)}}$$

$$= \frac{2}{x-1}$$

$\therefore$ there is a point of discontinuity (hole) at $(-1, -1)$

vertical asymptote at $x = 1$

horizontal asymptote at $y = 0$



1 mark for asymptotic behaviour at $x = 1$
1 mark for asymptotic behaviour at $y = 0$
1 mark for point of discontinuity (hole) at $(-1, -1)$ (½ mark for $x = -1$, ½ mark for $y = -1$)
½ mark for graph left of $x = 1$
½ mark for graph right of $x = 1$

4 marks

Question 26 P4

Justify why the binomial expansion of $\left(x + x^3\right)^7$ does not have a term containing x^{10}.

Solution

Method 1

$(x)^7,\ (x)^6\left(x^3\right)^1,\ (x)^5\left(x^3\right)^2,\ \ldots$

$x^7,\ x^9,\ x^{11},\ \ldots$

The exponents increase by 2.

Therefore x^{10} is not in the pattern.

1 mark for determining the pattern

1 mark for justification

2 marks

Method 2

$$x^{7-r}\left(x^3\right)^r = x^{10}$$

$$x^{7+2r} = x^{10}$$

$$7 + 2r = 10$$

$$2r = 3$$

$$r = \frac{3}{2}$$

½ mark for substitution

½ mark for solving for *r*

The binomial expansion does not contain x^{10} because the value of *r* must be a whole number.

1 mark for justification

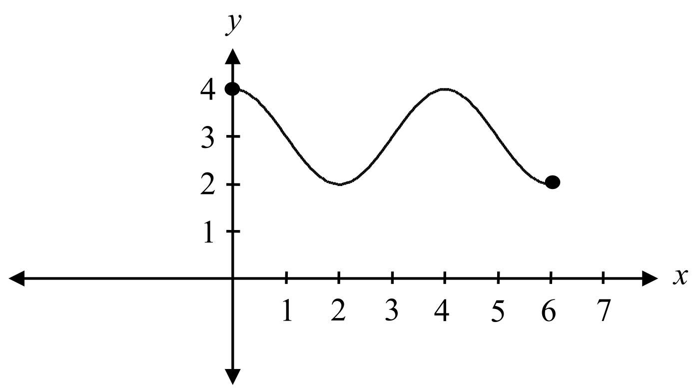
2 marks

Question 27 T4

Sketch the graph of $y = -\sin\left(\frac{\pi}{2}(x-1)\right) + 3$ over the domain $[0, 6]$.

Solution

$$\text{period} = \frac{2\pi}{\frac{\pi}{2}} = 4$$



1 mark for vertical reflection
1 mark for horizontal translation
1 mark for period
1 mark for vertical translation

4 marks

Note(s):

- Deduct ½ mark if the domain $[0, 6]$ is not completely sketched.

When $P(x) = 3x^4 - kx^3 + 5x - 14$ is divided by $(x + 2)$, the remainder is -8.

Determine the value of k.

Solution

Method 1

$x = -2$	½ mark for $x = -2$
$-8 = 3(-2)^4 - k(-2)^3 + 5(-2) - 14$	1 mark for remainder theorem
$-8 = 48 + 8k - 10 - 14$	
$-8 = 24 + 8k$	
$-32 = 8k$	
$k = -4$	½ mark for solving for k

2 marks

Method 2

-2	3	$-k$	0	5	-14
	$\downarrow$	-6	$2k + 12$	$-4k - 24$	$8k + 38$
	3	$-k - 6$	$2k + 12$	$-4k - 19$	$8k + 24$

½ mark for $x = -2$
1 mark for synthetic division (or any other equivalent strategy)

$-8 = 8k + 24$

$-32 = 8k$

$k = -4$

½ mark for solving for k

2 marks

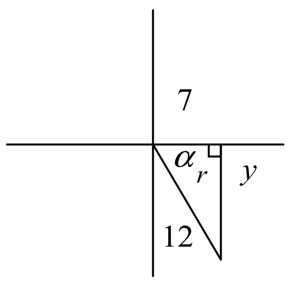
Given that $\cos\alpha = \frac{7}{12}$ where α is in quadrant IV, and $\sin\beta = \frac{3}{5}$ where β is in quadrant I, determine the exact value of:

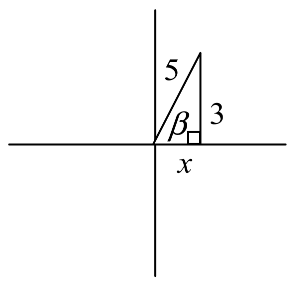
a) $\sin(\alpha - \beta)$

b) $\csc(\alpha - \beta)$

Solution

a)





$$12^2 - 7^2 = y^2 \qquad 5^2 - 3^2 = x^2$$
$$95 = y^2 \qquad 16 = x^2$$
$$\pm\sqrt{95} = y \qquad \pm 4 = x$$

½ mark for value of x
½ mark for value of y

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$= \left(\frac{-\sqrt{95}}{12}\right)\left(\frac{4}{5}\right) - \left(\frac{7}{12}\right)\left(\frac{3}{5}\right)$$
$$= \frac{-4\sqrt{95}}{60} - \frac{21}{60}$$
$$= \frac{-4\sqrt{95} - 21}{60}$$

½ mark for $\sin\alpha$
½ mark for $\cos\beta$
1 mark for substitution into correct identity

3 marks

b) $\csc(\alpha - \beta) = \dfrac{60}{-4\sqrt{95} - 21}$

1 mark

Note(s):

- accept any of the following values for x: $x = \pm 4$, or $x = 4$
- accept any of the following values for y: $y = \pm\sqrt{95}$, $y = \sqrt{95}$, or $y = -\sqrt{95}$

Describe the difference between the graph of $f(x)=\dfrac{7(x+2)}{x+2}$ and the graph of $g(x)=\dfrac{7(x-2)}{x+2}$ at $x=-2$.

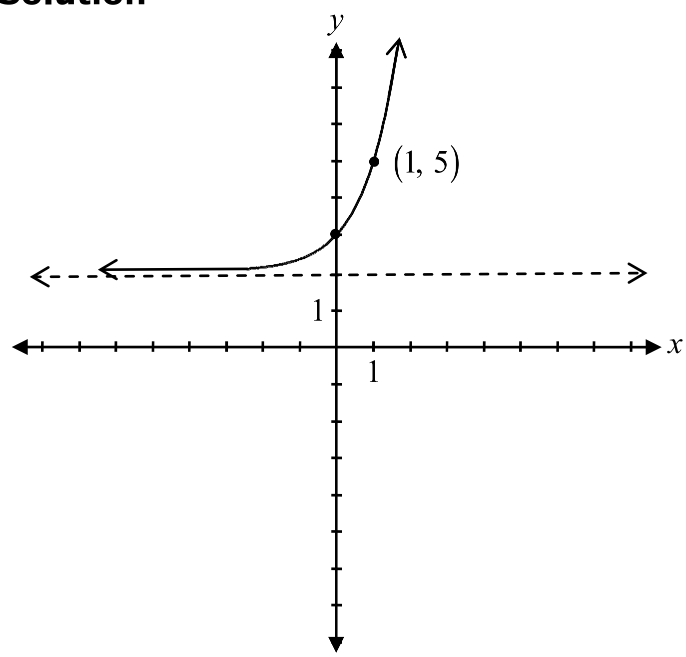
Solution

The graph of $f(x)=\dfrac{7(x+2)}{x+2}$ has a point of discontinuity and the graph of $g(x)=\dfrac{7(x-2)}{x+2}$ has an asymptote.

1 mark

Sketch the graph of $f(x) = 3^x + 2$.

Solution



1 mark for increasing exponential function
1 mark for asymptotic behaviour at $y = 2$

2 marks

Solve algebraically:

$${}_nC_3 = n - 2$$

Solution

$$\frac{n!}{(n-3)!3!} = n - 2$$ ½ mark for substitution

$$\frac{n(n-1)\cancel{(n-2)}\cancel{(n-3)!}}{\cancel{(n-3)!}3!} = \cancel{n-2}$$ 1 mark for factorial expansion

$$n(n-1) = 6$$ ½ mark for simplification of factorials

$$n^2 - n - 6 = 0$$

$$(n-3)(n+2) = 0$$

$$n = 3 \quad \cancel{n = -2}$$ ½ mark for rejecting extraneous root
½ mark for the value of n

3 marks

Describe the error that was made when solving the following equation:

$$\sin^2\theta + \sin\theta - 2 = 1$$

$$\sin^2\theta + \sin\theta = 3$$

$$\sin\theta(\sin\theta + 1) = 3$$

$\sin\theta = 3$ $\quad$ $\sin\theta + 1 = 3$

$\sin\theta = 2$

$\therefore$ No solution $\quad$ $\therefore$ No solution

Solution

The student did not apply the zero product principle before factoring.

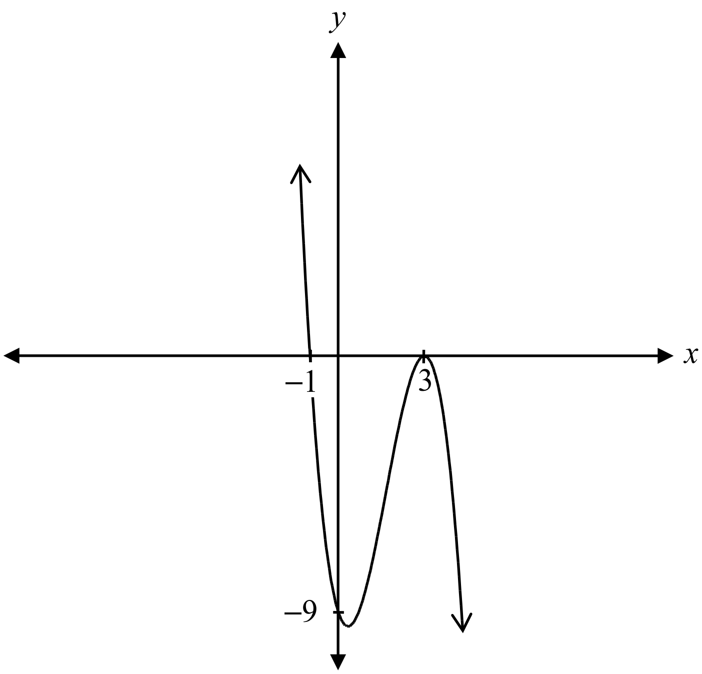
1 mark

Question 34 R12

Sketch the graph of the polynomial function with the following characteristics.

- a y-intercept of -9
- zeroes at -1 and 3
- the zero at -1 has a multiplicity of 1 and the zero at 3 has a multiplicity of 2

Solution



1 mark for x-intercepts

½ mark for y-intercept

1 mark for multiplicity (½ mark for multiplicity at $x = 3$, ½ mark for multiplicity at $x = -1$)

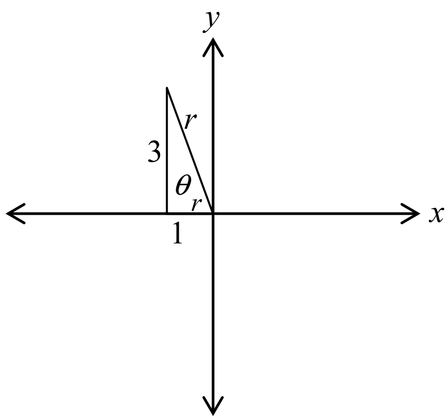
½ mark for shape of a cubic function

3 marks

Question 35 T2

Given $\cot\theta = -\frac{1}{3}$, where θ is in quadrant II, determine the exact value of $\sin\theta$.

Solution



$\cot\theta = \frac{x}{y}$

$r^2 = x^2 + y^2$

$r^2 = (-1)^2 + (3)^2$ ½ mark for substitution

$r^2 = 10$

$r = \pm\sqrt{10}$ ½ mark for solving for r

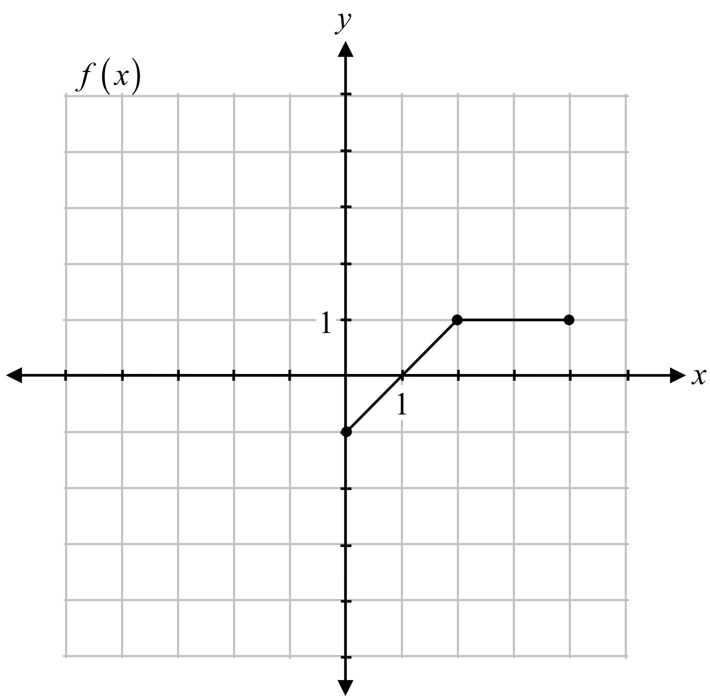
$\sin\theta = \frac{3}{\sqrt{10}}$ 1 mark for $\sin\theta$ (½ mark for the quadrant, ½ mark for the value)

2 marks

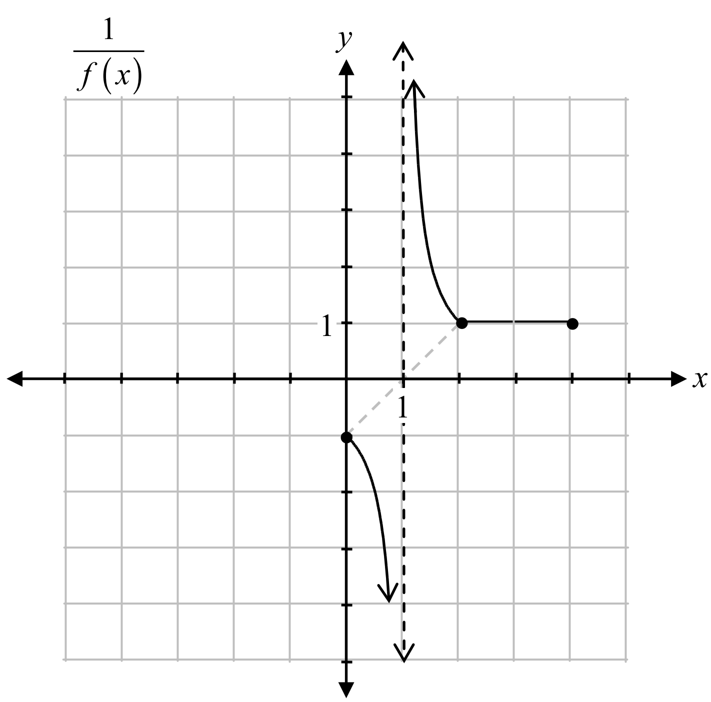
Note(s):

- accept any of the following values for r: $r = \pm\sqrt{10}$, $r = \sqrt{10}$

Given the function $f(x)$, sketch the graph of the reciprocal, $\frac{1}{f(x)}$.



Solution

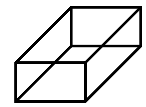


1 mark for asymptotic behaviour at $x = 1$
½ mark for graph left of vertical asymptote at $x = 1$
½ mark for graph right of vertical asymptote at $x = 1$

2 marks

Question 37 R12

The volume of a planter, in the shape of a rectangular prism, can be modelled by the polynomial function $V(x) = x^3 + 3x^2 - 34x + 48$.



Determine the factors of the function, $V(x)$, which represent possible dimensions of this planter.

Solution

$2^3 + 3(2)^2 - 34(2) + 48 = 0$

$\therefore x - 2$ is a factor — 1 mark for identifying one possible value of x

2	1	3	–34	48
	↓	2	10	–48
	1	5	–24	0

1 mark for synthetic division (or any other equivalent strategy)

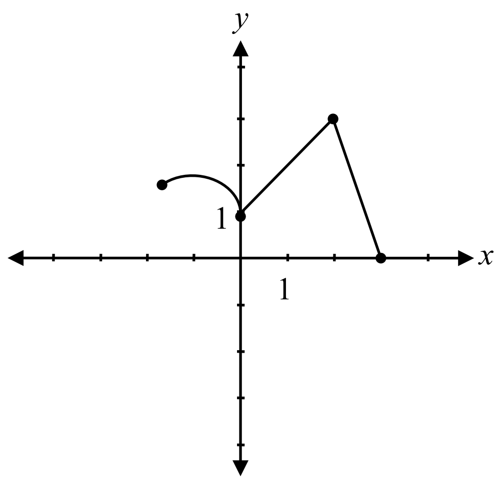
$V(x) = (x-2)\left(x^2 + 5x - 24\right)$

$V(x) = \underline{(x-2)(x+8)(x-3)}$ — 1 mark for identifying all factors

3 marks

Question 38 R6

Describe how to determine the range of the inverse of the following graph.



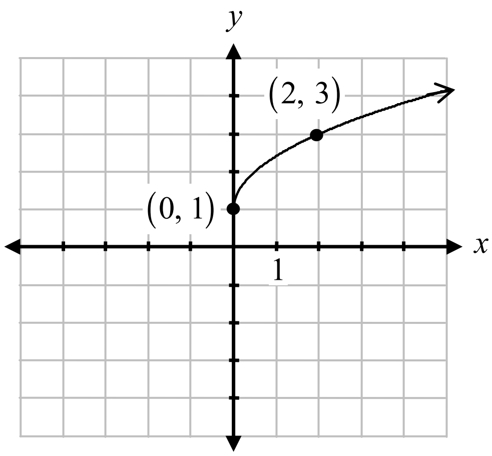
Solution

The domain of the graph becomes the range of the inverse.

1 mark

Sketch the graph of the function $y = \sqrt{2x} + 1$.

Solution



1 mark for shape of a radical function
1 mark for vertical translation
1 mark for horizontal compression

3 marks

Question 40 T4

Given the following characteristics of a sinusoidal function:

- an amplitude of 2
- a vertical translation down 3 units
- a period of $\frac{\pi}{4}$

a) Determine an equation of this sinusoidal function in the form $y = a\sin b(x - c) + d$.

b) Determine the range of this function.

Solution

a) $b = \dfrac{2\pi}{\frac{\pi}{4}}$

$b = 8$

$y = 2\sin(8x) - 3$

1 mark for the value of b
½ mark for amplitude
½ mark for vertical translation

2 marks

b) Range: $\{y \mid y \in \mathbb{R}, -5 \le y \le -1\}$

or

Range: $[-5, -1]$

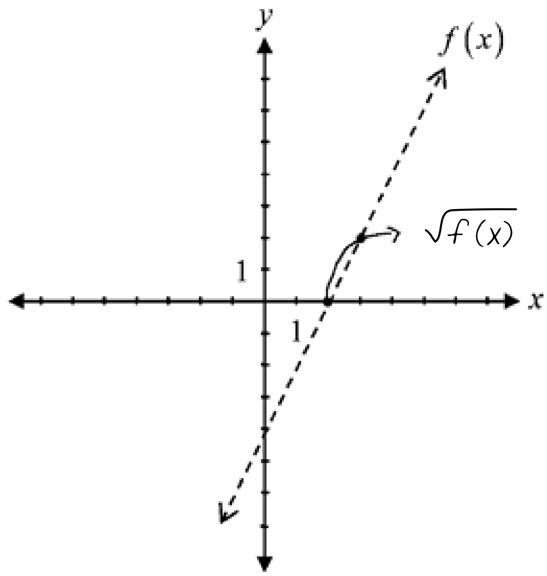
1 mark

Note(s):

- Other answers are possible for a).

Suah was given the graph of $f(x)$ and asked to graph $y = \sqrt{f(x)}$.

Her solution is given on the graph below.



Describe the error Suah made when sketching the graph of $y = \sqrt{f(x)}$.

Solution

Suah's graph did not cross the invariant point at $y = 1$.

1 mark

Question 42 R10

Solve:

$$9^{2x+1} = 27^x$$

Solution

$3^{2(2x+1)} = 3^{3x}$	1 mark for changing to a common base
$3^{4x+2} = 3^{3x}$	1 mark for exponent law (½ mark for each side)
$4x + 2 = 3x$	½ mark for equating exponents
$x = -2$	½ mark for solving for x

3 marks