LAF

Grade 12 Pre-Calculus Mathematics Notebook

Chapter 11

Permutations, Combinations, and Binomial Theorem

Outcomes: P1, P2, P3, P4

12P.P.1 Apply the fundamental counting principle to solve problems.

12P.P.2. Determine the number of permutations of n elements taken r at a time to solve problems.

12P.P.3. Determine the number of combinations of n different elements taken r at a time to solve problems.

12P.P.4. Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).

1. The Fundamental Counting Principle

You can choose one of 3 salads – Greek, Caesar, and Pasta – and one of 3 entrees – curry, hamburger, or sushi – from a menu. How many possible combo meals of one salad and one entree can you order?

We could make a complete list of all the possible meals...

But, this is a pain. What if there were 25 options for salad and 13 options for entree? What if there were also dessert options? We need a better method than making a list.

FUNDAMENTAL COUNTING PRINCIPLE: If one task can be performed in ***a*** ways and another task in ***b*** ways, then both tasks can be performed in ***a x b*** ways.

Example: If there are 7 possible salads, 9 possible entrees, and 5 possible desserts, there would be (7)(9)(5)= 315 possible meals. (Clearly, naming all 315 meals is impractical!)

Your Turn:

1. How many license plates are possible in Manitoba? (you need three digits and three letters)

26 x 26 x 26 x 10 x 10 x 10 = ______

(1st letter) (2nd letter) (3rd letter) (1st digit) (2nd digit) (3rd digit)

2. How many ways can a store manager hire for three positions given 12 candidates?

12 x 11 x 10 = ______

(1st Hire) (2nd Hire) (3rd Hire)

3. How many outfits can you order if you have 4 choices of pants, 5 shirts, and 6 pairs of socks?

$4 \cdot 5 \cdot 6$

4. How many 3 digit numbers can you make using the digits 1, 2, 3, 4, and 5 if:

(a) repetition is allowed

$5 \cdot 5 \cdot 5$

(b) repetition is NOT allowed

$5 \cdot 4 \cdot 3$

2. Factorial Notation!

6 people must be arranged in a row – how many different ways can this be done? (6)(5)(4)(3)(2)(1) = 720 ways.

We can re-write this as: $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ (We pronounce "6!" as "six factorial")

Find: 4! = $4 \cdot 3 \cdot 2 \cdot 1$

$= 24$

In general, we define factorial notation, for $n\epsilon\boldsymbol{N}$ as:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots\cdots \cdot 3 \cdot 2 \cdot 1$$

Oddly, we separately define 0! = 1. This is the **definition** – it is not based on a proof.

Your Turn:

1. Simplify:

a. $\frac{11!}{9!} = \frac{11 \cdot 10 \cdot 9!}{9!}$

$= 11 \cdot 10$

$= 110$

b. $\frac{5!}{7!} = \frac{5!}{7 \cdot 6 \cdot 5!}$

$= \frac{1}{42}$

c. $\frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!}$

$= n^2 - n$

d. $\frac{(n+1)!}{n!}$ $\frac{(n+1)n!}{n!} = n+1$

2. Prove that 5! – 3! = 19(3!)

$120 - 6$	$19(6)$
114	114

3. Solve: $\frac{n!}{(n-2)!} = 30$

$n(n-1) = 30$

$n^2 - n - 30 = 0$

$(n-6)(n+5) = 0$

$n = 6, \cancel{-5}$

3. Permutations

An arrangement of a set of objects in which the order of the objects is important is called a permutation. For example, "how many ways can you arrange 4 people sitting on a bench" is a permutation whereas "how many ways can you choose a group of 4 from the class?" is not.

a) Example: If you have 12 basketball players and 7 must be arranged on the bench, how many ways can you arrange the players on the bench?

$\underline{12} \cdot \underline{11} \cdot \underline{10} \cdot \underline{9} \cdot \underline{8}\ \underline{7}\ \underline{6} =$

b) all 12? $12! = 479001600$

c) Sada must be ~~at~~ ~~~~ ~~~~. in middle.

$\underline{11} \cdot \underline{10} \cdot \underline{9} \cdot \underline{1} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} = 332640$

d) Sada must be at end. Cases!

$\underline{1}\ \underline{11}\ \underline{10}\ \underline{9}\ \underline{8}\ \underline{7}\ \underline{6} = 332640$

$\underline{11}\ \underline{10}\ \underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{1} = 332640$

665280.

Ex2 How many 3 digit #'s can be formed using 0, 1, 2 3, 4 (no reps).

a) $\underline{4} \cdot \underline{4}\ \underline{3} = 48$

b) greater than. 3

$\underline{2} \cdot \underline{4} \cdot \underline{3} = 24$

3 or 4

c) Even

$\underline{4} \cdot \underline{3} \cdot \underline{1} = 12$ end in 0.

$\underline{3} \cdot \underline{3}\ \underline{2} = 18$ end in 2, 4

30.

4. Permutations with Identical Objects

Imagine that the letters of the word BOOK are written on tiles. We want to re-arrange the tiles. How many arrangements are possible?

$$\boxed{\frac{4!}{2!} = 12}$$

BOOK BOKO BKOO

OOBK OBOK yikes!

KOOB KBOO to long to LIST

Now that we've written them all out, we should notice that many of the arrangements are identical. For instance, KOBO* and KOBO look the same but I've switched the O's. So, although we have 4! permutations of the tiles, the two O's means that there are two non-distinct versions of each permutation. So, how many **distinct** permutations do we have? Well... 4!/2! = 12.

In general, a set of ***n*** objects with ***a*** objects that are identical can be arranged: $\frac{n!}{a!}$

In general, a set of ***n*** objects with ***a*** objects that are identical, ***b*** other identical objects and ***c*** other identical objects can be arranged:

$$\frac{n!}{a!b!c!} \text{ ways}$$

Your Turn:

1. How many different arrangements can you make using all the letters in M~~A~~NITOB~~A~~?

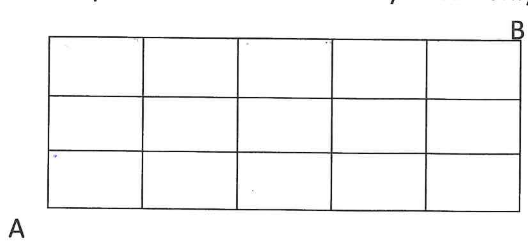
$$\frac{8!}{2!} =$$

2. How many different arrangements of all the numbers of 81818 can you make?

$$\frac{5!}{3!\,2!} =$$

3. How many different paths can you follow from A to B if you can only move up or right?

UUURRRRR



$$\frac{8!}{3!\,5!} =$$

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* Have you ever noticed the KOBO is just a permutation of BOOK? Clever marketing!

* deal with restrictions first.

5. Permutations with Objects Together

Example (From Textbook Page 522, Example 4):

Five people (A,B,C,D, and E) are seated on a bench. In how many ways can they be arranged if:

(a) E is seated in the middle?

$\underline{4} \cdot \underline{3} \cdot \underset{E}{\underline{1}} \cdot \underline{2} \cdot \underline{1} = 24.$

(b) A and B must sit together?

$\boxed{AB} \cdot C \quad D \quad E$

$4! \cdot 2! = 48$

(c) A and B must NOT sit together?

All possible − together.

$= 5! - 4! \cdot 2!$

$= 120 - 48$

$= 72.$

Your Turn:

You have to arrange 3 different calculus books, 5 different biology books, and 4 different economics books on a shelf.

(a) How many ways can you arrange the books if not restricted?

$12!$

(b) How many ways can you arrange the books if they must be arranged by subject area?

$3! \cdot 3! \cdot 5! \cdot 4!$ → economics.

"objects" calc bio

(c) How many ways can you arrange the books if all the Calculus books must go on the right end of the shelf?

$9! \; \underline{\underline{3!}}$

7. Combinations

A **combination** is a selection of objects where order doesn't matter.

Example: Form a group of three from students A, B, C, and D. The only combinations are ABC, ABD, ACD and BCD. Note that ABC is the same as ACB or BCA or BAC or CAB or CBA – those permutations all the same combination or group.

If you have ***n*** distinct items and ***r*** items are taken BUT NOT ARRANGED, then the number of **combinations** is:

$$_nC_r = \frac{_nP_r}{r!} = \frac{n!}{(n-r)!r!}$$

Note: $_nC_r$ is also written as $\binom{n}{r}$

Example 1: There are 14 females and 8 males in math class. You want to form a group of 5.

(a) How many groups can be formed?

$_{22}C_5 = 26334$

(b) How many groups can be formed with 2 females and 3 males?

$_{14}C_2 \cdot {_8C_3} = 91 \cdot 56 = 5096.$

(c) How many groups can be made with Joseph, one other male, and 3 females?

$1 \cdot {_7C_1} \cdot {_{14}C_3} = 2548$

Your Turn:

1. How many ways can a store manager hire for three identical positions given 12 candidates?

$_{12}C_3 = 220$

2. The chess club has 9 grade 10's, 5 grade 11's, and 14 grade 12's. The club's coach must select a team of 2 students from every grade to compete in a tournament. How many possible teams could she select?

$_9C_2 \cdot {_5C_2} \cdot {_{14}C_2}$

$= 36 \cdot 10 \cdot 91$

$= 32760.$

9. More Combinations

Example 2: On a test, you must answer 3 of the 5 short answer questions and 2 of the 6 essay questions. How many combinations of questions are possible?

$_5C_3 \cdot {_6C_2} = 150$

Example 3: On a test, you must answer at least 4 of the 5 short answer questions and exactly 2 of the 6 essay questions. How many combinations of questions are possible?

Case 1 (4 SA) $\quad _5C_4 \cdot {_6C_2} = 75$

Case 2 (5 SA) $\quad _5C_5 \cdot {_6C_2} = 15$

$= 90$

Your Turn:

A bag contains 6 red hockey pucks and 8 black hockey pucks. Assuming the pucks are distinguishable, draw 5 pucks from the bag. How many possible ways are there to have at least 3 red pucks?

3 pucks: $\quad _6C_3 \cdot {_8C_2} = 560$

4 pucks $\quad _6C_4 \cdot {_8C_1} = 120$

5 pucks $\quad _6C_5 \cdot {_8C_0} = 6$

$= 686$

~~6 pucks.~~

We only need 5!

Homework Practice Questions

Section 11.2 – Page 534, #1 – 8, 11, 13, 14, 17, C1, C3

8. Combination Arithmetic

Simplify: $\dfrac{{}_{n}C_{5}}{{}_{n-1}C_{3}}$

$$= \dfrac{\dfrac{n!}{(n-5)!\,5!}}{\dfrac{(n-1)!}{(n-1-3)!\,3!}}$$

$$= \frac{n!}{(n-5)!\,5!} \cdot \frac{(n-4)!\,3!}{\cancel{3!}\,(n-1)!}$$

$$= \frac{n\cancel{(n-1)!}(n-4)\cancel{(n-5)!}\,\cancel{3!}}{\cancel{(n-5)!}\;5\cdot 4\cdot \cancel{3!}\,\cancel{(n-1)!}}$$

$$= \frac{n(n-4)}{20}$$

Solve: $2({}_{n}C_{2}) = {}_{n+1}C_{3}$

$$\cancel{2}\left(\frac{n!}{(n-2)!\,\cancel{(2)}}\right) = \frac{(n+1)!}{(n+1-3)!\,3!}$$

$$\frac{3!\,\cancel{n!}\,\cancel{(n-2)!}}{\cancel{(n-2)!}} = \frac{(n+1)!}{n!}$$

$$6 = n+1$$

$$5 = n$$

10. Permutation Arithmetic

If you have ***n*** distinct items and ***r*** items are taken and arranged, then the number of permutations is:

$$_nP_r = \frac{n!}{(n-r)!}$$

For example, if you want to arrange 3 books in a display case and have 7 books to choose from, you can arrange the books:

$$_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 7 \cdot 6 \cdot 5 = 210\ ways$$

Your Turn:

1. Evaluate $_6P_4$

$\frac{6!}{(6-4)!}$

$\frac{6!}{2!}$

$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot \cancel{2}}{\cancel{2}}$

360

2. Solve: $_nP_2 = 30$

$\frac{n!}{(n-2)!} = 30$

$n(n-1) = 30$

$n^2 - n - 30 = 0$

$(n-6)(n+5) = 0$

$n = 6$ $\cancel{n = -5}$

3. Solve: $_nP_3 = 120$

$\frac{n!}{(n-3)!} = 120$

$n(n-1)(n-2) = 120$

$n = 6$

inspection!

note: if we say solve algebraically you can Not use inspection.