Chapter 6: TRIGONOMETRIC IDENTITIES

6.1 – Reciprocal, Quotient, and Pythagorean Identities

An **equation** like $\underbrace{\sin\theta = \frac{1}{2}}$ is true for very few values of θ, like $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$, or all coterminal angles with those angles. $[0, 2\pi]$

An **identity** like $\tan x = \frac{\sin x}{\cos x}$ is always true for all $\theta \in R$ (or nearly all).

permissible values

Recall the following identities:

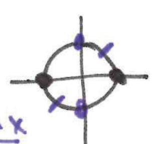
$\tan x = \frac{\sin x}{\cos x}$	$\csc x = \frac{1}{\sin x}$	$\sec x = \frac{1}{\cos x}$
$\cot x = \frac{\cos x}{\sin x}$	$\cot x = \frac{1}{\tan x}$	

Non-permissible values are any values that result in a denominator equal to 0.

Example #1

Determine the **non-permissible values** of the following identities, over the interval $\underbrace{0 \le x \le 2\pi}$

a) $\cot x = \frac{1}{\tan x}$ $\frac{\sin x}{\cos x}$ $\boxed{\frac{\cos x}{\sin x}}$



We know

$\tan x \neq 0$

$\frac{1}{\frac{\sin x}{\cos x}} = \frac{\cos x}{\sin x}$

$\boxed{x \neq 0, \pi, 2\pi}$

$\sin x \neq 0$

$x \neq 0, 2\pi, \pi$

"hidden denominator"

b) $\frac{\cot x}{(\tan x - 1)(\sin x)}$ $\frac{\cos x}{\sin x}$ ✓ $\frac{\sin x}{\cos x}$ **

We know

$\tan x - 1 \neq 0$

$\tan x \neq 1$

$\boxed{x \neq \frac{\pi}{4}, \frac{5\pi}{4}}$

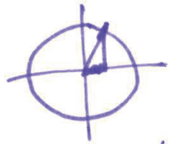
$\sin x \neq 0$

$\boxed{x \neq 0, \pi, 2\pi}$

we know

$\cos x \neq 0$

$\boxed{x \neq \frac{\pi}{2}, \frac{3\pi}{2}}$

MPC40S  Date: ____________________

Example #2

LHS RHS

Verify that the equation $\frac{\sec x}{\tan x + \cot x} = \sin x$ is true for the following values of x.

a) $x = 60°$ CHALLENGE

LHS

$$\frac{\sec 60°}{\tan 60° + \cot 60°}$$

$$\frac{2}{\sqrt{3} + \frac{\sqrt{3}}{3}}$$

$$\frac{2}{\frac{3\sqrt{3}}{3} + \frac{\sqrt{3}}{3}}$$

$$\frac{2}{\frac{3\sqrt{3}+\sqrt{3}}{3}}$$

$$\frac{2}{\frac{4\sqrt{3}}{3}}$$

$$\frac{2}{1}\left(\frac{3}{4\sqrt{3}}\right)$$

$$\frac{6}{4\sqrt{3}}$$

$$\frac{6}{4\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$\frac{6\sqrt{3}}{4(3)}$$

$$\frac{6\sqrt{3}}{12}$$

$$\frac{\sqrt{3}}{2}$$

RHS

$$\sin 60°$$

$$\frac{\sqrt{3}}{2}$$

LHS ≠ RHS

∴ 60° is a SOLUTION!

b) $x = \frac{3\pi}{4}$ →

LHS

$$\frac{\sec \frac{3\pi}{4}}{\tan \frac{3\pi}{4} + \cot \frac{3\pi}{4}}$$

$$\frac{-\frac{2}{\sqrt{2}}}{-1 - 1}$$

$$\frac{-\frac{2}{\sqrt{2}}}{-2}$$

$$\frac{-2}{\sqrt{2}} \cdot \frac{-1}{2}$$

$$\frac{1}{\sqrt{2}}$$

RHS

$$\sin \frac{3\pi}{4}$$

$$\frac{1}{\sqrt{2}}$$

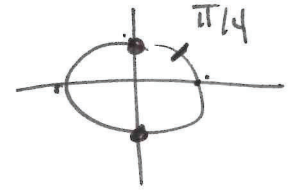
LHS = RHS

∴ the equation is true for $x = \frac{3\pi}{4}$

c) State the **non-permissible values** of the above identity over the interval $0 \leq x \leq 2\pi$

$$\frac{\sec x}{\tan x + \cot x} = \sin x$$

$$\frac{1}{\cos x} \qquad \frac{\sin x}{\cos x} \qquad \frac{\cos x}{\sin x}$$



We know

$\cos x \neq 0$ $\qquad$ $\sin x \neq 0$

$x \neq \frac{\pi}{2}, \frac{3\pi}{2}, 0, \pi, 2\pi$

$\tan x + \cot x \neq 0$

$\tan x \neq -\cot x$

won't happen!

d) npv's over the reals?

$x \neq \frac{\pi}{2} + \pi n$ $\qquad$ $x \neq \pi n.$

$n \in I$

$n \in Z$

Pythagorean Identities

From the unit circle, we know that $\cos\theta = x$ and $\sin\theta = y$.

We also know that the equation of the unit circle is $\boxed{x^2 + y^2 = 1}$

Therefore, upon substitution,

1). $(\cos\theta)^2 + (\sin\theta)^2 = 1$

$\boxed{\sin^2\theta + \cos^2\theta = 1}$

Dividing Equation 1) by $\cos^2\theta$ results in:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \rightarrow$$

2). $\tan^2\theta + 1 = \sec^2\theta$

$\frac{\cancel{\sin\theta}}{\cancel{\tan\theta}}$

$\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right)$

Dividing Equation 1) by $\sin^2\theta$ results in:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \rightarrow$$

3). $1 + \cot^2\theta = \csc^2\theta$

We can manipulate these equations to create other equations.

$\sin^2\theta + \cos^2\theta = 1$ ✓	$\tan^2\theta + 1 = \sec^2\theta$ ✓	$1 + \cot^2\theta = \csc^2\theta$ ✓
$\cos^2\theta = 1 - \sin^2\theta$	$\tan^2\theta = \sec^2\theta - 1$	$\cot^2\theta = \csc^2\theta - 1$
$\sin^2\theta = 1 - \cos^2\theta$	$\cancel{1 \neq \sec^2\theta + \tan^2\theta}$ $1 = \sec^2\theta - \tan^2\theta$	$\cancel{1 = \cot^2\theta + \csc^2\theta}$ $1 = \csc^2\theta - \cot^2\theta$

Note: These identities are only true when the trigonometric functions are squared. (i.e. $\sin\theta + \cos\theta \neq 1$)

Example #3

Simplify the following expressions completely.

a) $\frac{\tan\theta\cos\theta}{\sec\theta\cot\theta}$

$$\frac{\frac{\sin\theta}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos\theta}}{1}}{\frac{1}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos\theta}}{\sin\theta}}$$

$$\frac{\frac{\sin\theta}{1}}{\frac{1}{\sin\theta}}$$

$$\frac{\sin\theta}{1}\left(\frac{\sin\theta}{1}\right)$$

$$\sin^2\theta$$ ✓

$$1-\cos^2\theta$$

b) $\frac{\cot x}{\csc x\cos x}$

$$\frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}\cdot\frac{\cos x}{1}}$$

$$\frac{\frac{\cos x}{\sin x}}{\frac{\cos x}{\sin x}}$$

$$\frac{\cos x}{\sin x}\cdot\frac{\sin x}{\cos x}$$

$$1$$

c) $\frac{\sec^2\theta\cos\theta}{\csc\theta}$

$$\frac{\frac{1}{\cos^2\theta}\cdot\frac{\cos\theta}{1}}{\frac{1}{\sin\theta}}$$

$$\frac{\frac{1}{\cos\theta}}{\frac{1}{\sin\theta}}$$

$$\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{1}$$

$$\frac{\sin\theta}{\cos\theta}$$

$$\tan\theta$$