Grade 12
Pre-Calculus Mathematics
[MPC40S]

Chapter 6

Trigonometric Identities

Outcomes

T5, T6

12P.T.5. Solve, algebraically first and second degree trigonometric equations with the domain expressed in degrees and radians.

12P.T.6. Prove trigonometric identities, using
- Reciprocal identities
- Quotient identities
- Pythagorean identities
- Sum or difference (restricted to sine, cosine, and tangent)
- Double-angle identities (restricted to sine, oscine, and tangent.

Chapter 6 – Homework

Section	Page	Questions

Chapter 6: TRIGONOMETRIC IDENTITIES

6.1 – Reciprocal, Quotient, and Pythagorean Identities

An **equation** like $\sin\theta = \frac{1}{2}$ is true for very few values of θ, like $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$, or all coterminal angles with those angles. $[0, 2\pi]$

An **identity** like $\tan x = \frac{\sin x}{\cos x}$ is always true for all $\theta \in R$ (or nearly all).

permissible values

Recall the following identities:

$\tan x = \frac{\sin x}{\cos x}$	$\csc x = \frac{1}{\sin x}$	$\sec x = \frac{1}{\cos x}$
$\cot x = \frac{\cos x}{\sin x}$	$\cot x = \frac{1}{\tan x}$	

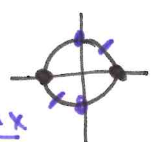
Non-permissible values are any values that result in a denominator equal to 0.

Example #1

Determine the **non-permissible values** of the following identities, over the interval $0 \le x \le 2\pi$

$\frac{\cos x}{\sin x}$

a) $\cot x = \frac{1}{\tan x}$ → $\frac{\sin x}{\cos x}$



$\frac{1}{\frac{\sin x}{\cos x}} = \frac{\cos x}{\sin x}$

We know

$\tan x \neq 0$

$\boxed{x \neq 0, \pi, 2\pi}$

$\sin x \neq 0$

$x \neq 0, 2\pi, \pi$

"hidden denominator"

b) $\frac{\cot x}{(\tan x - 1)(\sin x)}$

$\frac{\cos x}{\sin x}$ ✓

$\frac{\sin x}{\cos x}$ **

We know

$\tan x - 1 \neq 0$

$\tan x \neq 1$

$\boxed{x \neq \frac{\pi}{4}, \frac{5\pi}{4}}$

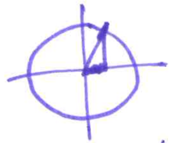
$\sin x \neq 0$

$\boxed{x \neq 0, \pi, 2\pi}$

we know

$\cos x \neq 0$

$\boxed{x \neq \frac{\pi}{2}, \frac{3\pi}{2}}$

MPC40S  Date: ____________________

Example #2 LHS RHS

Verify that the equation $\frac{\sec x}{\tan x+\cot x}=\sin x$ is true for the following values of x.

a) $x = 60°$ CHALLENGE

LHS

$\frac{\sec 60°}{\tan 60° + \cot 60°}$

$\frac{2}{\sqrt{3}+\frac{\sqrt{3}}{3}}$

$\frac{2}{\frac{3\sqrt{3}}{3}+\frac{\sqrt{3}}{3}}$

$\frac{2}{\frac{3\sqrt{3}+\sqrt{3}}{3}}$

$\frac{2}{\frac{4\sqrt{3}}{3}}$

$\frac{2}{1}\left(\frac{3}{4\sqrt{3}}\right)$

$\frac{6}{4\sqrt{3}}$

$\frac{6}{4\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$

$\frac{6\sqrt{3}}{4(3)}$

$\frac{6\sqrt{3}}{12}$

$\frac{\sqrt{3}}{2}$

RHS

$\sin 60°$

$\frac{\sqrt{3}}{2}$

LHS ≠ RHS

∴ 60° is a solution!

b) $x = \frac{3\pi}{4}$ → ✓

LHS

$\frac{\sec\frac{3\pi}{4}}{\tan\frac{3\pi}{4}+\cot\frac{3\pi}{4}}$

$\frac{-\frac{2}{\sqrt{2}}}{-1-1}$

$\frac{-\frac{2}{\sqrt{2}}}{-2}$

$\frac{-2}{\sqrt{2}}\cdot\frac{-1}{2}$

$\frac{1}{\sqrt{2}}$

RHS

$\sin\frac{3\pi}{4}$

$\frac{1}{\sqrt{2}}$

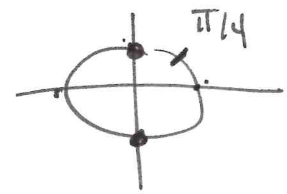
LHS = RHS

∴ the equation is true for $x = \frac{3\pi}{4}$

c) State the **non-permissible values** of the above identity over the interval $0 \le x \le 2\pi$

$$\frac{\sec x}{\tan x + \cot x} = \sin x$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$



We know

$\cos x \neq 0$ $\sin x \neq 0$

$x \neq \frac{\pi}{2}, \frac{3\pi}{2}, 0, \pi, 2\pi$

$\tan x + \cot x \neq 0$

$\tan x \neq -\cot x$

won't happen!

d) npv's over the reals?

$x \neq \frac{\pi}{2} + \pi n$ $x \neq \pi n.$

$n \in I$

$n \in Z$

Pythagorean Identities

From the unit circle, we know that $\cos\theta = x$ and $\sin\theta = y$.

We also know that the equation of the unit circle is $\boxed{x^2 + y^2 = 1}$

Therefore, upon substitution, 1). $(\cos\theta)^2 + (\sin\theta)^2 = 1$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

Dividing Equation 1) by $\cos^2\theta$ results in:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \rightarrow$$

2). $\tan^2\theta + 1 = \sec^2\theta$

$\cancel{\sin\theta}$

$\cancel{\tan\theta}$

$\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right)$

Dividing Equation 1) by $\sin^2\theta$ results in:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \rightarrow$$

3). $1 + \cot^2\theta = \csc^2\theta$

We can manipulate these equations to create other equations.

$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$1 + \cot^2\theta = \csc^2\theta$
$\cos^2\theta = 1 - \sin^2\theta$	$\tan^2\theta = \sec^2\theta - 1$	$\cot^2\theta = \csc^2\theta - 1$
$\sin^2\theta = 1 - \cos^2\theta$	$\cancel{1 = \sec^2\theta + \tan^2\theta}$	$\cancel{1 = \cot^2\theta + \csc^2\theta}$
	$1 = \sec^2\theta - \tan^2\theta$	$1 = \csc^2\theta - \cot^2\theta$

Note: These identities are only true when the trigonometric functions are squared. (i.e. $\sin\theta + \cos\theta \neq 1$)

Example #3

Simplify the following expressions completely.

a) $\frac{\tan\theta\cos\theta}{\sec\theta\cot\theta}$

$$\frac{\frac{\sin\theta}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos\theta}}{1}}{\frac{1}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos\theta}}{\sin\theta}}$$

$$\frac{\frac{\sin\theta}{1}}{\frac{1}{\sin\theta}}$$

$$\frac{\sin\theta}{1}\left(\frac{\sin\theta}{1}\right)$$

$$\sin^2\theta$$

$$1-\cos^2\theta$$

b) $\frac{\cot x}{\csc x\cos x}$

$$\frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}\cdot\frac{\cos x}{1}}$$

$$\frac{\frac{\cos x}{\sin x}}{\frac{\cos x}{\sin x}}$$

$$\frac{\cos x}{\sin x}\cdot\frac{\sin x}{\cos x}$$

$$1$$

c) $\frac{\sec^2\theta\cos\theta}{\csc\theta}$

$$\frac{\frac{1}{\cos^2\theta}\cdot\frac{\cos\theta}{1}}{\frac{1}{\sin\theta}}$$

$$\frac{\frac{1}{\cos\theta}}{\frac{1}{\sin\theta}}$$

$$\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{1}$$

$$\frac{\sin\theta}{\cos\theta}$$

$$\tan\theta$$

Chapter 6: TRIGONOMETRIC IDENTITIES

6.3 – Proving Identities Part 1

To **PROVE AN IDENTITY** means to manipulate both sides of an equation independently until they are identical.

Helpful Hints

- Rewrite all functions in terms of $\sin\theta$ and $\cos\theta$
- Factor
- Multiply by a form of 1 $\left(\text{for example: } \frac{\sin\theta}{\sin\theta}\right)$
- Simplify
- Expand
- Multiply by the conjugate $(1+\cos x \to 1-\cos x)$

Things to Remember

- Identities are different from equations. Keep side separate!
- Use the correct variable $(\theta, x, \alpha, \beta, \text{etc.})$
- Left Hand Side = Right Hand Side (LHS = RHS)
- Show all of your work (**marks are allocated for work shown**)
- Do not invent new math. It is better to not arrive at the same answer than to make something up!
- Work vertically ↓

Example #1

Prove the following identities for all permissible values of the variable.

a) $(\sin x)(\sec x)(\cot x) = 1$

LHS	RHS
$\cancel{\sin x}\left(\frac{1}{\cancel{\cos x}}\right)\left(\frac{\cancel{\cos x}}{\cancel{\sin x}}\right)$	
1	1

LHS = RHS

b) $\frac{\cos^2 x}{\cot x} = \sin x \cos x$

LHS	RHS
$\dfrac{\cos^2 x}{\frac{\cos x}{\sin x}}$	
$\cos^2 x\left(\frac{\sin x}{\cos x}\right)$	
$\cos x \sin x$	$\cos x \sin x$

LHS = RHS

c) $\frac{\sec x}{\tan x+\cot x} = \sin x$

LHS	RHS
$\dfrac{\frac{1}{\cos x}}{\frac{(\sin x)\sin x}{(\sin x)\cos x} + \frac{\cos x\,(\cos x)}{\sin x\,(\cos x)}}$	
$\dfrac{\frac{1}{\cos x}}{\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}}$	
$\dfrac{\frac{1}{\cos x}}{\frac{1}{\sin x \cos x}}$	
$\frac{1}{\cos x}\left(\frac{\sin x \cos x}{1}\right)$	
$\sin x$	$\sin x.$

LHS = RHS.

MPC40S $\sin^2\theta + \cos^2\theta = 1$ Date: ______________

d) $\underbrace{1 - \cos^2\theta}_{} = \cos^2\theta \tan^2\theta$

LHS	RHS
$\sin^2\theta$	$\frac{\cancel{\cos^2\theta}}{1} \cdot \frac{\sin^2\theta}{\cancel{\cos^2 x}}$
	$\sin^2\theta$
	LHS = RHS.

e) $\frac{\sin\alpha}{1-\cos\alpha} = \frac{1+\cos\alpha}{\sin\alpha}$

LHS	RHS
$\frac{\sin\alpha}{1-\cos\alpha} \cdot \frac{1+\cos\alpha}{1+\cos\alpha}$	
$\frac{\sin\alpha(1+\cos\alpha)}{1+\cos\alpha-\cos\alpha-\cos^2\alpha}$	$\frac{1+\cos\alpha}{\sin\alpha}$
$\frac{\sin\alpha(1+\cos\alpha)}{1-\cos^2\alpha}$	
$\frac{\sin\alpha(1+\cos\alpha)}{\sin^2\alpha}$	LHS = RHS
$\frac{1+\cos\alpha}{\sin\alpha}$	

f) $\cos^4\theta - \sin^4\theta = 1 - 2\sin^2\theta$

LHS	RHS
$(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$	
$\cos^2\theta - \sin^2\theta$	
$1 - \sin^2\theta - \sin^2\theta$	
$1 - 2\sin^2\theta$	$1 - 2\sin^2\theta$

LHS = RHS.

Chapter 6: TRIGONOMETRIC IDENTITIES

6.2 – Sum, Difference, and Double-Angle Identities

Along with the Pythagorean Identities, we have additional identities we can work with. They are the Sum and Difference Identities and the Double Angle Identities.

Sum and Difference Identities

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

Double Angle Identities

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\cos 2\alpha = 1-2\sin^2\alpha$

$\cos 2\alpha = 2\cos^2\alpha - 1$

$\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

With the help of these identities, we can now we can find the coordinates of any point on the unit circle, provided it is a sum or difference of our special angels.
$\left(\frac{\pi}{3},\frac{\pi}{4},\frac{\pi}{6}\text{ family members}\right)$

Example #1

Simplify the following expressions and give the **exact value**.

a) $\cos\frac{\pi}{3}\cos\frac{\pi}{12}+\sin\frac{\pi}{3}\sin\frac{\pi}{12}$

$\cos(\alpha-\beta)$

$\cos\left(\frac{\pi}{3}-\frac{\pi}{12}\right)$

$\cos\left(\frac{4\pi}{12}-\frac{\pi}{12}\right)$

$\cos\left(\frac{3\pi}{12}\right)$

$\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$=\frac{\sqrt{2}}{2}$

b) $\cos^2(47°)+\sin^2(47°)$ $=1$

c) $1-2\sin^2\frac{\pi}{6}$

$=\cos 2\alpha$

$=\cos \frac{2}{1}\left(\frac{\pi}{6}\right)$

$=\cos\frac{\pi}{3}$

$=\frac{1}{2}$

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We can use these formulas to determine the exact value of a trigonometric ratio that is not on the unit circle.

Example #2

$30°, 45°, 60°, 120°$
$135°$

Determine the **exact value** of each expression below.

a) $\sin\frac{7\pi}{12}$

$\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$

$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$

$= \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$

$\frac{\pi}{6}$ $\frac{\pi}{3}$ $\frac{\pi}{4}$ $\frac{\pi}{2}$

$\uparrow$ $\uparrow$

$\frac{2\pi}{12}$ $\boxed{\frac{4\pi}{12} \quad \frac{3\pi}{12}}$ $\frac{6\pi}{12}$

b) $\cos 195°$

$\cos(60° + 135°)$

$= \cos 60° \cos 135° - \sin 60° \sin 135°$

$= \frac{1}{2}\left(-\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{-\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$

$= \frac{-\sqrt{2}-\sqrt{6}}{4}$

We can simplify expressions using identities.

Example #3

Consider the expression $\frac{1-\cos 2\theta}{\sin 2\theta}$

a) Determine the **non-permissible values** for this expression between $[0, 2\pi]$

$\sin 2\theta \neq 0$

$2\sin\theta\cos\theta \neq 0$

$\sin\theta \neq 0$ $\quad\quad$ $\cos\theta \neq 0$

$\boxed{\theta \neq 0, \pi, 2\pi \quad\quad \theta \neq \frac{\pi}{2}, \frac{3\pi}{2}}$

b) **Simplify** the expression to one of the three primary trigonometry functions.

$$\frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$$

$$\frac{1-1+2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$\frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$\frac{\sin\theta}{\cos\theta}$$

$$\tan\theta$$

Example #4

If $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \frac{3}{5}$, determine the **exact value** of $\cos 2\theta$. $= 1 - 2\sin^2\theta$

$= 1 - \left(\frac{3}{5}\right)^2$

$= 1 - \frac{9}{25}$

$= \frac{25}{25} - \frac{9}{25}$

$= \frac{16}{25}$

Example #5

If $\cos\alpha = -\frac{4}{5}$ where $\frac{\pi}{2} \le \alpha \le \pi$ and $\cos\beta = -\frac{12}{13}$ where $\pi \le \beta \le \frac{3\pi}{2}$, determine the **exact value** of $\cos(\alpha - \beta)$.

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

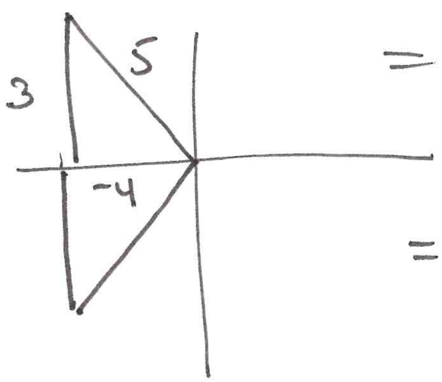
$$= \frac{-4}{5}\left(\frac{-12}{13}\right) + \sin\alpha\sin\beta$$

$$= \frac{-4}{5}\left(\frac{-12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{-5}{13}\right)$$

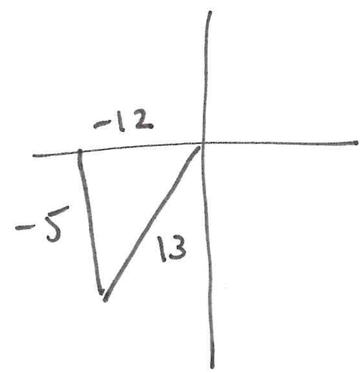
$$= \frac{48}{65} - \frac{15}{65}$$

$$= \frac{33}{65}$$

α



$\sin\alpha = \frac{3}{5}$



$a^2+b^2=c^2$

$\sin\beta = \frac{-5}{13}$

Chapter 6: TRIGONOMETRIC IDENTITIES

6.3 – Proving Identities Part 2

Example #1

Prove the following identities for all permissible values of the variable.

a) $\cot\theta - \csc\theta = \frac{\cos 2\theta - \cos\theta}{\sin 2\theta + \sin\theta}$

LHS	RHS
$\frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}$	$\frac{2\cos^2\theta - 1 - \cos\theta}{2\sin\theta\cos\theta + \sin\theta}$
$\frac{\cos\theta - 1}{\sin\theta}$	$\frac{2\cos^2\theta - \cos\theta - 1}{\sin\theta(2\cos\theta + 1)}$
	$\frac{(2\cos\theta + 1)(\cos\theta - 1)}{\sin\theta(2\cos\theta + 1)}$
	$\frac{\cos\theta - 1}{\sin\theta}$

LHS = RHS

b) $\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

LHS	RHS

$$\frac{\frac{\cos^2\theta}{\cos^2\theta}\frac{1}{1} - \frac{\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta}\frac{1}{1} + \frac{\sin^2\theta}{\cos^2\theta}}$$

$$\frac{\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}}$$

$$\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta} \cdot \frac{\cos^2\theta}{1}$$

$\cos 2\theta$

$$\cos^2\theta - \sin^2\theta$$

$$\cos 2\theta$$

LHS = RHS

Chapter 6: TRIGONOMETRIC IDENTITIES

6.4 – Solving Trigonometric Equations Using Identities

We will be solving trigonometric equations just like in the past, but now we need to make substitutions using trigonometric identities.

Example #1

Solve the following trigonometric equation $\sin 2\theta = \sin\theta$ over the interval $[0, 2\pi]$

$\sin 2\theta - \sin\theta = 0$

$2\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta(2\cos\theta - 1) = 0$

$\sin\theta = 0$ $\quad\quad$ $\cos\theta = \frac{1}{2}$

$\theta = 0, \pi, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}$

Example #2

Solve the following trigonometric equation over the interval $[0, 2\pi)$.

$\cos x = 0 \quad \sin x \neq 0$

$x \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

$\tan x \cos x \sin x \cot x \csc x - 1 = 0$

$\cancel{\frac{\sin x}{\cos x}} \cdot \cos x \cdot \sin x \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} - 1 = 0$

$\cos x - 1 = 0$

$\cos x = 1$

$x = \cancel{0}$

No solution

Example #3

a) Determine the **non-permissible values** of the following equation over the interval $[0, 2\pi)$.

$$\tan\theta = 2\sin\theta$$

$$\frac{\sin\theta}{\cos\theta} = 2\sin\theta$$

$\cos\theta \neq 0$

$\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}$

b) **Solve** the above trigonometric equation over the given interval

$\sin\theta = 2\sin\theta\cos\theta$

$0 = 2\sin\theta\cos\theta - \sin\theta$

$0 = \sin\theta(2\cos\theta - 1)$.

$\sin\theta = 0$

$\theta = 0, \pi$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Example #4

Solve the following trigonometric equations over the interval $[0, 2\pi]$.

a) $\cos 2x + 1 = \cos x$

$2\cos^2 x - 1 + 1 = \cos x$

$2\cos^2 x - \cos x = 0$

$\cos x\ (2\cos x - 1) = 0$

$\cos x = 0 \qquad \cos x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, \frac{5\pi}{3}$

b) $1 - \cos^2 x = 3\sin x - 2$

$\sin^2 x = 3\sin x - 2$

$\sin^2 x - 3\sin x + 2 = 0$

$(\sin x - 2)(\sin x - 1) = 0$

$\sin x = 2 \qquad \sin x = 1$

No solution $\qquad x = \frac{\pi}{2}$

c) $2\sin x = 7 - 3\csc x$

$\sin x \neq 0$

$x \neq 0, \pi, 2\pi$

$2\sin x = 7 - \frac{3}{\sin x}$

$2\sin^2 x = 7\sin x - 3$

$2\sin^2 x - 7\sin x + 3 = 0$

$(2\sin x - 1)(\sin x - 3) = 0$

$\sin x = \frac{1}{2}$

$\sin x = 3$

No solution.

$x = \frac{\pi}{6}, \frac{5\pi}{6}$