SLOT 6

Grade 12
Pre-Calculus Mathematics
[MPC40S]

Chapter 3

Polynomial Functions

Outcomes

R11, R12

12P.R.11. Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree $\leq$ 5with integral coefficients).

11P.R.12 Graph and analyze polynomial functions (limited to polynomial functions of degree $\leq$ 5).

Chapter 3: POLYNOMIAL FUNCTIONS

3.1 – Characteristics of Polynomial Functions

Polynomial Function: A function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n = is a whole # (exponents)

x = is a variable

a_n and a_0 = can be any value.

Example

The following are examples of polynomial functions.

$f(x) = 3x - 5$

$g(x) = x^2 + 3x - 17$

$h(x) = x^3 + 2x^2 + x - 2$

$y = x^5 + 7x^3 - 1$

Example #1

Identify the functions that are not polynomials and state why.

a) $g(x) = \sqrt{x} + 5$ $y = x^{1/2} + 5$ → Radical function.

b) $y = |x|$ Not! Absolute value function!

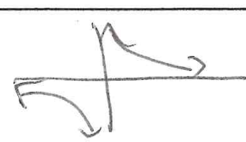
c) $f(x) = 3x^4$

d) $y = x^{\frac{1}{8}} - 7$ Not! exponent not a whole #.

e) $y = 2x^3 + 3x^2 - 4x - 1$

f) $h(x) = \frac{1}{x}$ NOT! The exponent is not a whole #.

$h(x) = x^{-1}$



End Behaviour: the behavior of the y values as $|x|$ gets infinitely ~~big~~ Big!

Degree: The power with the highest exponent!

Constant Term: The term without a variable.

Leading Coefficient: the coefficient in front of the term with the highest exponent.

Example #2

For each polynomial function, state the **degree**, the **leading coefficient**, and the **constant term** of each polynomial function.

a) $y = 3x^2 - 2x^5 + 4$

- Degree 5 - Leading Coefficient -2 - Constant Term 4

b) $y = -4x^3 - 4x + 3$

- Degree 3 - Leading Coefficient -4 - Constant Term 3

c) $f(x) = 3x - 5$

- Degree 1 - Leading Coefficient 3 - Constant Term -5

d) $f(x) = -6x^4 - 2x + 0$

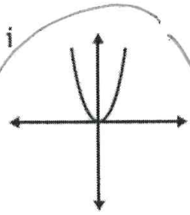
- Degree 4 - Leading Coefficient -6 - Constant Term 0

Compare the graphs of **even degree** and **odd degree** functions. How does the leading term affect the general behaviour of the graph?

the L.C. affects the end behavior!

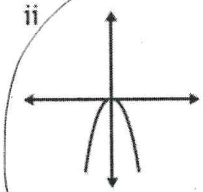
a) The equations and graphs of several **even-degree** polynomials are shown below. Study these graphs and generalize the end behaviour of **even-degree** polynomials.

i



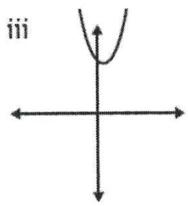
$f(x) = x^2$
quadratic

ii



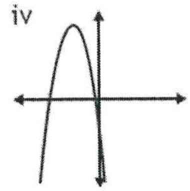
$f(x) = -x^2$
quadratic

iii



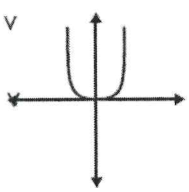
$f(x) = x^2 - x + 6$
quadratic

iv



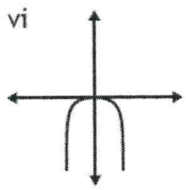
$f(x) = -x^2 - 8x - 7$
quadratic

v



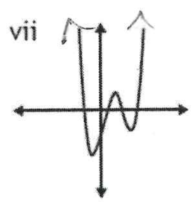
$f(x) = x^4$
quartic

vi



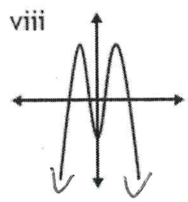
$f(x) = -x^4$
quartic

vii



$f(x) = x^4 - 4x^3 + x^2 + 7x - 3$
quartic

viii

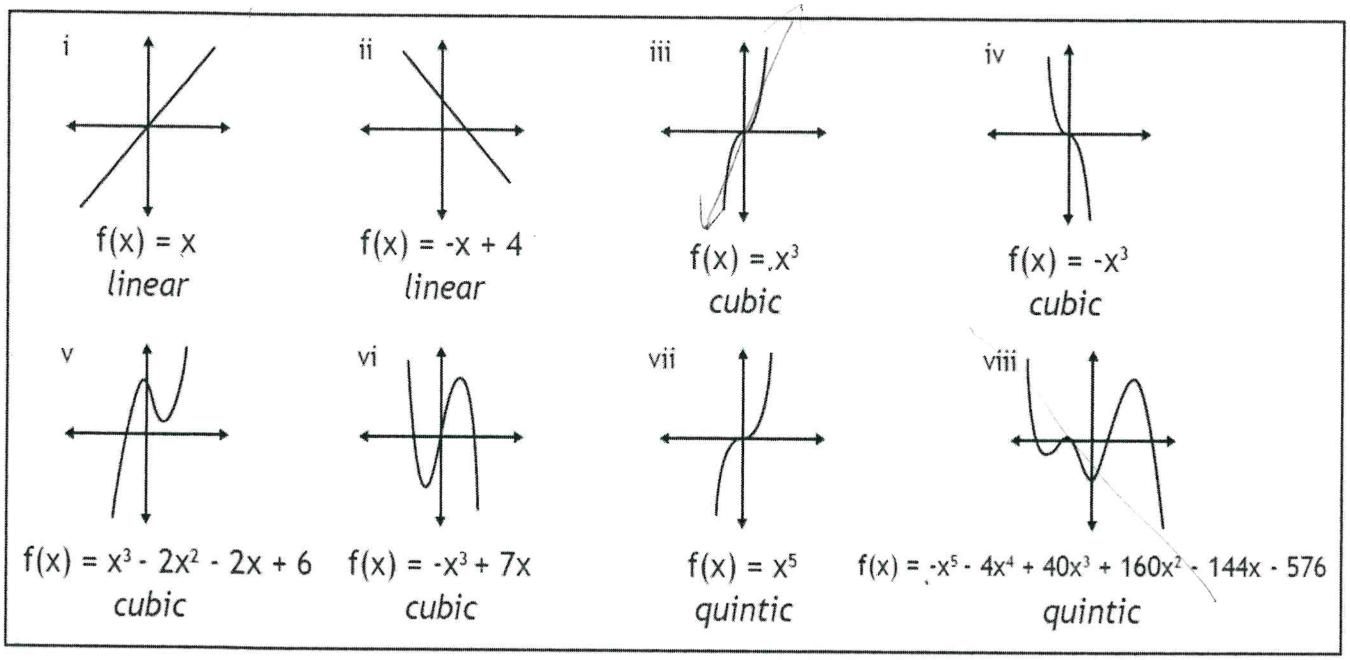


$f(x) = -x^4 + 7x^2 - 5$
quartic

What do you notice about the **end behaviour** of an **even-degree** polynomial when...

The **leading coefficient** is positive?	The **leading coefficient** is negative?
Even degree	Even
→ open up into	→ open down into
Q I and Q II	Q III and Q IV

b) The equations and graphs of several **odd-degree** polynomials are shown below. Study these graphs and generalize the end behaviour of **odd-degree** polynomials.



What do you notice about the **end behaviour** of an **odd-degree** polynomial when…

The **leading coefficient** is positive?	The **leading coefficient** is negative?
go down into Q III	go up into Q II
and up into Q I	and down into Q IV

Notes
- The graph of a polynomial function must be smooth and continuous
- The graph has **at most** $(n - 1)$ turning points
- The function has **at most** n roots (x – intercepts)
- All polynomial functions have y – intercept at a_0, the constant term of the function

Example #3

Match the following polynomials with its corresponding graph.

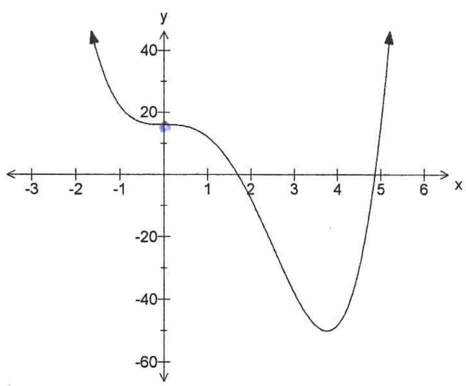
1.) $f(x) = 2x^3 - 4x^2 + x + 2$

2.) $g(x) = -x^4 + 10x^2 + 5x - 6$

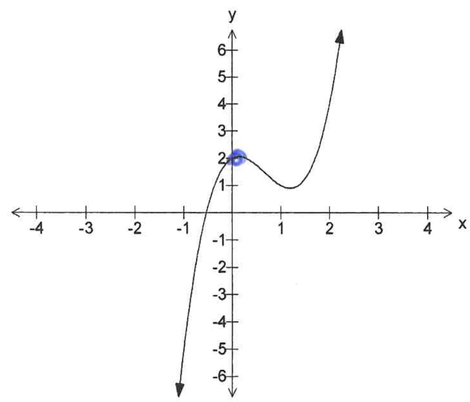
3.) $h(x) = -2x^5 + 5x^3 - x + 1$

4.) $p(x) = x^4 - 5x^3 + 16$

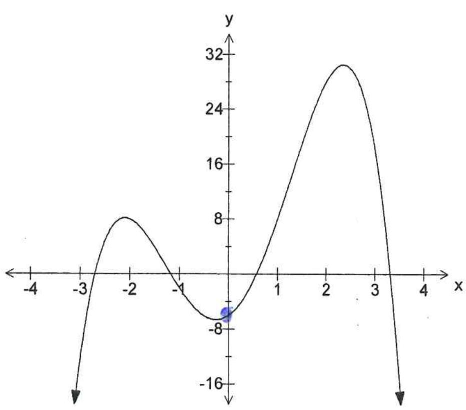
a) Answer: 4



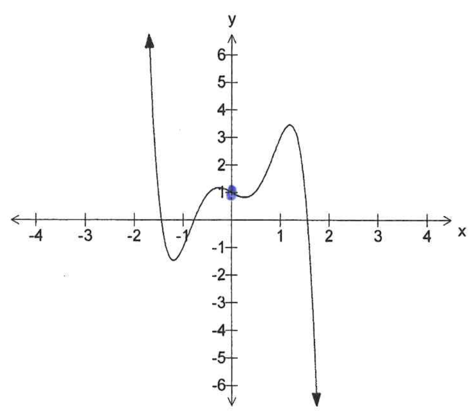
b) Answer: 1



c) Answer: 2



d) Answer: 3



MPC40S Date: ______________ **R11**

Chapter 3: POLYNOMIAL FUNCTIONS

3.2 – The Remainder Theorem

We are going to learn how to divide a polynomial by a binomial.

Dividend = Polynomial

Divisor = Binomial (x – a)

Quotient = Answer

Long Division (Method #1)

Example #1

Divide the following expression $\frac{x^2+8x+15}{x+3}$

$x \neq -3$

$$
\begin{array}{r}
x + 5 \\
x+3 \,\big)\, \overline{x^2 + 8x + 15} \\
-(x^2 + 3x) \qquad \\
5x + 15 \\
-(5x + 15) \\
\hline
0
\end{array}
$$

① Divide first term in the dividend by the first term in the divisor.
② Multiply this by each term in the divisor
③ pull down the next ~~...~~ term and repeat!

After you divide, your answer can be written in two forms:

1) $\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{remainder}}{\text{Divisor}}$ **OR** 2) Dividend = (Divisor)(Quotient) + remainder

Answer to the above example:

$\frac{x^2+8x+15}{x+3} = x+5 + \frac{0}{x+3}$ $x^2+8x+15 = (x+5)(x+3) + 0$

Note: Since the remainder is 0, this tells us that $(x+3)$ is a **factor** of the polynomial $x^2+8x+15$

Note: The restriction on the variable is $x \neq -3$ since division by 0 is not defined.

Synthetic division is an alternate form of long division that we can use to divide polynomials.

This type of division uses only the coefficients of each equation.

Steps:
1. Rearrange the equation in descending order
2. Write **only** the coefficients of the polynomial. If any are **missing**, fill in their spot with a **zero**.
3. Bring down the first coefficient.
4. Multiply by the divisor.
5. Add that number to the second coefficient.
6. Repeat steps 4-6 until there are no more coefficients to bring down.
7. Write the resulting numbers as the coefficients of a new polynomial. The last number will be the remainder.

Example #2

Divide the following expression $\frac{x^2+8x+15}{x+3}$

$x-a$
$x=a$ divisor

-3	1	8	15
$\times$		-3	-15
$+$	1	5	0

$$\frac{x^2+8x+15}{x+3} = 1x^1 + 5 + \frac{0}{x+3}$$

Example #3

$0x^2$

Divide $2x^4 + 2x^3 - x + 4$ by $x + 2$

$$\begin{array}{r|rrrrr}
-2 & 2 & 2 & 0 & -1 & 4 \\
\times & \downarrow & -4 & 4 & -8 & 18 \\
\hline
+ & 2 & -2 & 4 & -9 & 22
\end{array}$$

$$\frac{2x^4+2x^3-x+4}{x+2} = 2x^3 - 2x^2 + 4x - 9 + \frac{22}{x+2}$$

Example #4

$0x$

Divide $(2x^3 + 5x^2 + 9) \div (x + 3)$

$$\begin{array}{r|rrrr}
-3 & 2 & 5 & 0 & 9 \\
 & \downarrow & -6 & 3 & -9 \\
\hline
 & 2 & -1 & 3 & 0
\end{array}$$

$$\frac{2x^3+5x^2+9}{x+3} = 2x^2 - x + 3 + \frac{0}{x+3}$$

or

$$2x^3+5x^2+9 = (2x^2-x+3)(x+3) + 0$$

Example #5

Divide $x^3 - 2x^2 + 6x - 12$ by $x - 1$

$$\begin{array}{c|cccc} 1 & 1 & -2 & 6 & -12 \\ & & 1 & -1 & 5 \\ \hline & 1 & -1 & 5 & -7 \end{array}$$

$$\frac{x^3 - 2x^2 + 6x - 12}{x - 1} = x^2 - x + 5 \quad \frac{-7}{x - 1}$$

Example #6

~~Divide~~ $x^3 - 7x^2 - 6x + 72$ by $x - 4$

What is the remainder?

Substitute $x = 4$

$4^3 - 7(4)^2 - 6(4) + 72$

$64 - 7(16) - 24 + 72$

$64 - 112 - 24 + 72$

0

Chapter 3: POLYNOMIAL FUNCTIONS
3.3 – The Factor Theorem

The **Factor Theorem** tells us whether or not the **divisor** $(x - a)$ is a **factor** of the dividend.

If there is no remainder (i.e. the remainder = 0), then the divisor is a factor.

The **Factor Theorem** states that $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$

Example #1

a) Determine whether or not $x + 2$ is a factor of $P(x) = x^3 + 4x^2 + x - 6$

$P(-2) = (-2)^3 + 4(-2)^2 - 2 - 6$

$P(-2) = -8 + 4(4) - 2 - 6$

$P(-2) = -8 + 16 - 8$

$P(-2) = 0$ $\therefore$ $(x+2)$ is a factor!

b) If $x + 2$ is a factor, completely **factor** $P(x) = x^3 + 4x^2 + x - 6$

Divide $P(x)$ by $(x+2)$

-2	1	4	1	-6
$\times$	$\downarrow$	-2	-4	6
$+$	1	2	-3	0

$P(x) = (x+2)(x^2 + 2x - 3)$

$P(x) = (x+2)(x-1)(x+3)$

Example #2

Completely **factor** $P(x) = x^3 - 7x + 6$

To do this, we must find the factors of $P(x) = x^3 - 7x + 6$.
Let's use the Remainder Theorem.

Try $(x+1)$ $P(-1) = (-1)^3 - 7(-1) + 6$

$P(-1) =$ ____________________

$P(-1) =$ 0

Try $(x+2)$ $P(-2) = (-2)^3 - 7(-2) + 6$

$P(-2) =$ ____________________

$P(-2) \neq$ 0

Try $(x-3)$ $P(3) = (3)^3 - 7(3) + 6$

$P(3) =$ ____________________

$P(3) =$ 0

Try $(x-1)$ $P(1) = (1)^3 - 7(1) + 6$

$P(1) =$ ____________________

$P(1) =$ 0

$p(x) = (x-1)(x-2)(x+3)$.

There must be an easier way than randomly guessing infinitely many times...

Integral Zero Theorem

Expand the following expression:

$(x-1)(x+2)(x-5) =$ $x^3 - 4x^2 - 7x + 10$

Note: The **factors** of the polynomial are $(x-1)$, $(x+2)$ and $(x-5)$

The **zeros** of the polynomial are 1, –2 and 5

Note: When we multiply all of the factors, the constant is +10, which means that only factors of 10 can be factors of the polynomial.

This is known as the **Integral Zero Theorem**

> The **Integral Zero Theorem** states that if $(x-a)$ is a factor of the polynomial function $P(x)$ with integral coefficients, then a is a factor of the constant term of $P(x)$.

Example #3

a) Find all of the **possible** integral zeros of the following polynomial:

$f(x) = x^3 - 3x^2 - 6x + 8 \rightarrow$ $\pm 1, \pm 2, \pm 4, \pm 8$

factors of 8

b) Completely **factor** the polynomial above.

$f(x) = x^3 - 3x^2 - 6x + 8$

① Create a list of possible integral zeroes.

② Use the Factor theorem to find one that works.

$f(2) = 2^3 - 3(2)^2 - 6(2) + 8$
$f(2) = 8 - 3(4) - 12 + 8$
$f(2) = 8 - 12 - 12 + 8$
$f(2) \neq 0$

$f(-2) = 0$ $\therefore$ $\underline{x+2}$ is a factor.

③ Divide!

④ Factor!

$$\begin{array}{r|rrrr} -2 & 1 & -3 & -6 & 8 \\ & & -2 & 10 & -8 \\ \hline & 1 & -5 & 4 & 0 \end{array}$$

$P(x) = (x+2)(x^2 - 5x + 4)$

$P(x) = (x+2)(x-4)(x-1)$

Example #4

a) Find all of the **possible** integral zeros of the following polynomial:

$$f(x) = 2x^3 - 3x^2 - 8x + 12 \rightarrow \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$$

b) Completely **factor** the polynomial above.

$f(-2) = 0$

$f(x) = (x-2)(2x-3)(x+2).$

This zero is a must!

c) Determine the x-intercepts ~~**zeros**~~ of $f(x)$.

$0 = (x-2)(2x-3)(x+2)$

$x = 2, \frac{3}{2}, -2.$

Example #5

a) Completely **factor** $P(x) = x^4 - 5x^3 + 2x^2 + 20x - 24$

Two times the work!

$P(x) = (x-3)(x-2)^2(x+2)$

b) Determine the **zeros** of $P(x)$.

$0 = (x-3)(x-2)^2(x+2)$

$x = 3, 2, -2$

The Remainder Theorem

The **remainder theorem** allows us to obtain the **value of the remainder** without actually dividing.

When $P(x)$ is divided by $(x-a)$ the remainder is $P(a)$

Example #7

Use the **remainder theorem** to determine the **remainder** when the polynomial $P(x) = x^3 - 5x^2 - 17x + 21$ is divided by the following binomials.

Verify your solution using either long division or synthetic division.

a) $x + 1$

$P(-1) = (-1)^3 - 5(-1)^2 - 17(-1) + 21$

$P(-1) = -1 - 5(1) + 17 + 21$

$P(-1) = -1 - 5 + 17 + 21$

$P(-1) = 32$

b) $x - 1$

$P(1) = (1)^3 - 5(1)^2 - 17(1) + 21$

$P(1) = 0$

$\therefore$ $x-1$ is a factor of $P(x)$